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Zero-Dissipative Trigonometrically Fitted Hybrid Method for Numerical Solution of Oscillatory Problems

(Kaedah Hibrid Penyuaian Trigonometri Lesapan-Sifar untuk Penyelesaian Berangka Masalah Berayun)

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ABSTRACT

In this paper, an improved trigonometrically fitted zero-dissipative explicit two-step hybrid method with fifth algebraic order is derived. The method is applied to several problems where by the solutions are oscillatory in nature. Numerical results obtained are compared with existing methods in the scientific literature. The comparison shows that the new method is more effective and efficient than the existing methods of the same order.

Keywords: Dispersion; hybrid method; oscillatory problems; oscillatory solution; trigonometrically fitted

ABSTRAK

Dalam kertas ini, suatu penyuaian trigonometri lesapan sifar kaedah hibrid dua langkah penambahbaikan peringkat kelima diterbitkan. Kaedah ini digunakan untuk beberapa masalah yang penyelesaiannya berayun. Keputusan berangka yang diperoleh dibandingkan dengan kaedah sedia ada dalam maklumat saintifik. Perbandingan tersebut menunjukkan kaedah yang yang baharu ini adalah lebih efektif dan cekap berbanding kaedah sedia ada dengan peringkat yang sama.

Kata kunci: Kaedah hibrid; masalah berayun; penyelesaian berayun; penyuaian trigonmetri; serakan

INTRODUCTION

In the last few decades, there has been a growing interest in the research of new numerical techniques for approximating the solution of second order initial value problem of the form

$$y'' = f(x, y), y(x_0) = y_0, y'(x_0) = y'_0, \quad (1)$$

which is independent on y' explicitly. This type of problem arises in different fields of science and engineering, which includes quantum mechanics, celestial mechanics, molecular dynamics, quantum chemistry, astrophysics, electronics and semi-discretizations of wave equation.

Numerous numerical methods have been derived for approximating the solutions of (1), some of which are Runge-Kutta methods, Runge-Kutta Nyström methods and linear multistep methods. For the Runge-Kutta methods and other related methods specifically derived for approximating the solutions of first order initial value problems (IVPs), the second order IVPs need to be transformed to a system of first order IVPs so that the methods can be applied. In the quest for methods that best approximate the solution of (1), many authors considered different modifications on Runge-Kutta methods, multistep methods and Runge-Kutta Nyström methods, such work can be seen in Al-Khasawneh et al. (2007); Butcher (2008) Dormand et al. (1987), Franco (2006), Ming et al. (2012), Senu et al. (2010) and Simos (2002). Hybrid type methods related to multistep methods have been proposed by many authors for approximating the solutions of (1) (Simos 2012, 1999; Tsitouras 2002).

Franco (2006) proposed explicit two-step hybrid methods up to algebraic order six with less computational cost by using the algebraic order conditions of two-step hybrid methods developed by Coleman (2003). In furtherance to this, Ahmad et al. (2013) proposed semi-implicit hybrid methods up to algebraic order five. In the numerical integration of oscillatory or periodic problems, consideration of dispersion and dissipation errors is very important. The higher is the order of the errors for a method, the more efficient it is for solving oscillatory problems. Highest order of dispersion achieved for constant coefficients two-step hybrid method in the literature is eight (Franco 2006). Dispersive of order infinity two-step hybrid methods which coefficients are variables depending on the frequency of the problem to be solved were derived by Ahmad et al. (2013), Dizicheh et al. (2012) and Fang and Wu (2007).

In this paper we present a trigonometrically fitted two-step hybrid method based on zero-dissipative five stage fifth order hybrid method derived by Yusuf Dauda (2014). Again, the coefficient of the method depends on the frequency of the problem to be solved; hence the frequency of the problems must be known in advance.

The issue of how to choose the frequencies in trigonometrically fitted techniques if the frequency is not known in advance is very difficult. This problem has been considered by Vanden Beghe et al. (2001) whereby the frequency is tuned at every step by analyzing the local error. In their work, Ramos and Vigo-Aguiar (2010) conjectured that the frequency depends on the numerical method, the problem to be solved together with the initial value as well as the stepsize chosen.

Hybrid method can be written as follows:

$$Y_i = (1 + c_i)\, y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, Y_j),\ i = 1, \ldots, s, \tag{2}$$

$$Y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{j=1}^{s} b_i f(t_n + c_i h, Y_j), \tag{3}$$

where y_{n+1} and are approximations to $y(x_{n+1})$ and $y(x_n)$ respectively.

The paper is organized as follows: The dispersion and dissipation analysis of two-step hybrid methods are given in the next section. In section that follows, we look at the stability aspect of the method followed by the derivation of the method next. Numerical results and discussion is given in the next section and finally the conclusion is given in the last section.

Dispersion and Dissipation Analysis

In this section we discuss the dispersion and dissipation of explicit hybrid methods. Consider the homogeneous test problem,

$$y''(x) = -\lambda^2 y(x) \text{ for } v > 0.$$

By replacing $f(x, y) = -\lambda^2 y$ into (2) and (3) we have:

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} - h^2 \sum_{j=1}^{s} a_{ij}\, \lambda^2 y_j,\ \ i = 1, \ldots, s, \tag{4}$$

$$y_{n+1} = 2y_n - y_{n-1} - h^2 \sum_{j=1}^{s} b_i \lambda^2 y_i. \tag{5}$$

Alternatively (4) and (5) can be written in vector form as:

$$\boldsymbol{Y} = (\boldsymbol{e} + \boldsymbol{c})y_n - cy_{n-1} - z^2 \boldsymbol{AY},$$

$$y_{n+1} = 2y_n - y_{n-1} - z^2 \boldsymbol{b}^T \boldsymbol{Y},$$

where $z = hv$ $\boldsymbol{Y} = (Y_1, \ldots, Y_s)^T, = (c_1, \ldots, c_s)^T,$

$$\boldsymbol{e} = (1, \ldots, 1)^T,\ \boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{pmatrix},$$

and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix},\ z = \lambda h.$

From (4) we obtain:

$$\boldsymbol{Y} = (\boldsymbol{I} + z^2\boldsymbol{A})^{-1}(\boldsymbol{e} + \boldsymbol{c})y_n - (\boldsymbol{I} + z^2\boldsymbol{A})^{-1}cy_{n-1}, \tag{6}$$

We substitute (6) into (5) and obtained:

$$y_{n+1} = (2 - z^2\boldsymbol{b}^T(\boldsymbol{I} + z^2\boldsymbol{A})^{-1}(\boldsymbol{e} + \boldsymbol{c})\, y_n - (1 - z^2\boldsymbol{b}^T(\boldsymbol{I} + z^2\boldsymbol{A})^{-1}c)y_{n-1}. \tag{7}$$

Rewrite (7) and the following recursion is obtained,

$$y_{n+1} - T(z^2)y_n + D(z^2)y_{n-1} = 0, \tag{8}$$

where

$$T(z^2) = 2 - z^2\boldsymbol{b}^T(\boldsymbol{I} + z^2\boldsymbol{A})^{-1}(\boldsymbol{e} + \boldsymbol{c})$$

$$D(z^2) = 1 - z^2\boldsymbol{b}^T(\boldsymbol{I} + z^2\boldsymbol{A})^{-1}\boldsymbol{c}.$$

The solution of the difference equation (8), (Van der Houwen & Sommeijer 1987) is given by,

$$y_n\ 2|c||\rho|^m\cos(\omega + n\varphi), \tag{9}$$

and the solution to the test problem is,

$$y(x_n) = 2|\delta|\cos(\psi + nz), \tag{10}$$

where ρ is the amplification factor, φ is the phase and ω, δ, ψ are real constants. The definition formulated by Van de houwen and Sommeijer (1987) follows immediately.

Definition 1: *From (9) and (10), the quantity $R(z) = z - \varphi$ is called dispersion error of the method. The method is said to have dispersion error of order q if $R(z) = O(z^{q+1})$. Furthermore, the quantity $S(z) = 1 - |\rho|$ is called dissipation error of the method. And the method is said to be dissipative of order r if $S(z) = O(z^{r+1})$.*

Stability Analysis

In the analysis of stability of explicit hybrid methods, the test equation $y''(x) = -\lambda^2 y(x)$ need to be solved using the method. Rewriting equation (8) gives the stability polynomial of the method as:

$$\xi^2 - T(z^2)\, \xi + D(z^2) = 0.$$

The magnitude of the dispersion and dissipation errors is an important feature for solving second order IVPs (1) with periodic or oscillating solutions. In such problems, it is also important that the numerical solution obtained from the difference in (8) should be periodic as it is the exact solution of the test equation $y''(x) = -\lambda^2 y(x)$. This is true only if its coefficients satisfy the following conditions (Franco 2006),

$$D(z^2) \equiv 1, |T(z^2)| < 2, \forall\ z^2 \in (0, z_p^2).$$

where $(0, z_p^2)$ is known as the interval of periodicity of the method. With this, the method is said to be zero-dissipative since $S(z) = 0$, which implies that the order of dissipation of the method is infinity. On the other hand, if the order of dissipation is finite that is $S(z) \neq 0$, then absolute stability of the method is guaranteed on the following condition;

$$|D(z^2| < 1, |T(z^2| < 1 + D(z^2), \forall\ z^2 \in (0, z_s^2),$$

and the interval $(0, z_s^2)$ is known as the interval of absolute stability of the method. The method derived in this paper is periodic because of the zero-dissipative property of the method.

Methods

Trigonometrically fitted methods are generally derived to exactly approximate the solution of the IVPs whose solutions are linear combination of the functions:

$$\{x^j e^{\alpha x}, x^j e^{-\alpha x}\},$$

where α can be complex or a real number. Suppose $G(x) = e^{-i\alpha x}$, were $i = \sqrt{-1}$ is an imaginary unit, is the solution of (1). Then applying (2) to $G(x)$ generates the following recursive relations (Fang & Wu 2007),

$$\cos(c_i z) = (1 + c_i) - c_i\cos(z) - z^2 \sum_{i=1}^{i-1} a_{ij} \cos(c_j z),$$
$$\sin(c_i z) = c_i\sin(z) - z^2 \sum_{i=1}^{i-1} a_{ij} \sin(c_j z),$$
$$\cos(c_i z) = 1 - \frac{1}{2} z^2 \sum_{i=1}^{m} b_i \cos(c_i z),$$
$$\sum_{i=1}^{m} b_i \sin(c_i z) = 0.$$

The relations replace the equations of algebraic order conditions of two-step hybrid method, which can be solved to give the coefficients of a particular method based on existing coefficients for solving problem of the form (1).

The algebraic order conditions of two-step hybrid methods derived by (Coleman 2003) is given:

Order 1: $\Sigma\, b_i = 1$

Order 2: $\Sigma\, b_i c_i = 0$

Order 3: $\Sigma\, b_i c_i^2 = \frac{1}{6}, \sum b_i a_{ij} = \frac{1}{12},$

Order 4: $\Sigma\, b_i c_i^3 = 0, \sum b_i c_i a_{ij} = \frac{1}{12},$

$\Sigma\, b_i a_{ij} c_j = 0,$

Order5: $\Sigma\, b_i c_i^4 = \frac{1}{15}, \sum b_i c_i^2 a_{ij} = \frac{1}{30},$

$\Sigma\, b_i c_i a_{ij} c_j = -\frac{1}{60},$

All subscripts run to m or less. To derive the fifth order five stage method, we substitute $m = 5$ in the recursive relations to get,

$$\cos(c_3 z) - 1 - c_3 + c_3\cos(z) + z^2(a_{31}\cos(z) + a_{32}) = 0 \tag{11}$$

$$\sin(c_3 z) - c_3\sin(z) - z^2(a_{31}\sin(z)) = 0 \tag{12}$$

$$\cos(c_4 z) - 1 - c_4 + c_4\cos(z) + z^2(a_{41}\cos(z) + a_{42}\cos(c_3 z)) = 0 \tag{13}$$

$$\sin(c_4 z) - c_4\sin(z) + z^2(-a_{41}\sin(z) + a_{43}\sin(c_3 z)) = 0 \tag{14}$$

$$\cos(c_5 z) - 1 - c_5 + c_5\cos(z) + z^2(a_{51}\cos(z) + a_{52} + a_{53}\cos(c_3 z) + a_{54}\cos(c_5 z) = 0 \tag{15}$$

$$\sin(c_5 z) - c_5\sin(z) + z^2(-a_{51}\sin(z) + a_{53}\sin(c_3 z) + a_{54}\sin(c_5 z)) = 0 \tag{16}$$

$$2\cos(z) - 2 + z^2(b_1\cos(z) + b_2 + b_3\cos(c_3 z) + b_4\cos(c_4 z) + b_5\cos(c_5 z)) = 0 \tag{17}$$

$$-b_1\sin(z) + b_3\sin(c_3 z) + b_4\sin(c_4 z) + b_5\sin(c_5 z) = 0 \tag{18}$$

Equations (11)-(18) are the equations of order conditions for five stage fifth order trigonometrically fitted hybrid method that replaced equations of algebraic order conditions of the original method. To obtain the coefficients of the method we solve the system of eight equations, (11)-(18), together with some additional equations from algebraic order conditions of order five, namely;

$$b_1 + b_2 + b_3 + b_4 + b_5 = 1, \tag{19}$$

$$b_1 + b_3 c_3^4 + b_4 c_4^4 + b_5 c_5^4 = \tfrac{1}{15}, \tag{20}$$

$$-b_1 + b_3 c_3^3 + b_4 c_4^3 + b_5 c_5^3 = 0, \tag{21}$$

These sums up to eleven equations with seventeen unknowns' parameters. The equations are solved in terms of six free parameters ($c_3 = 1, c_4 = \frac{1}{2}, c_5 = -\frac{511}{1000}, a_{43} = 0,\ a_{53} = \frac{42413378651}{40000000000000}, a_{54} = 0$), whose values are obtained from the coefficients of HM5(5,6,∞), presented in Table 1 as obtained from (Dauda Yusuf 2014). Equations (19)-(21) are chosen to augment the updated (17)-(18) so that b_i are not taken as free parameters.

$$b_1 = -\frac{1}{6}\frac{M_1}{M_2},\ b_2 = -\frac{1}{3}\frac{M_3}{M_4},$$

$$b_3 = -\frac{1}{6}\frac{M_5}{M_6},\ b_4 = -\frac{1600}{3}\frac{M_7}{M_8},$$

$$b_5 = \frac{M_9}{M_{10}}.$$

TABLE 1. Coefficient of HM5(5,6,

-1	0				
0	0	0			
1	0	1	0		
$\frac{1}{2}$	$\frac{1}{16}$	$\frac{5}{16}$	0	-6	
$-\frac{511}{1000}$	$-\frac{209298067349}{400000000000}$	$-\frac{16643665561}{2000000000000}$	$-\frac{42413378651}{4000000000000}$	0	0
	$\frac{89}{5868}$	$\frac{1355}{3066}$	$\frac{311}{18132}$	$\frac{800}{3033}$	$\frac{100000000000}{381720407859}$

HM5(5,6,): zero dissipative fifth order five stage sixth order dispersive hybrid method derived in Yusouf Dauda (2014)

$$
\begin{aligned}
M_1 = {} & 3000000000000 \cos(z)\sin\left(\frac{511}{1000}z\right) \\
& -809403552846 \sin(z) \\
& + 9677616366768 \sin\left(\frac{1}{2}z\right) \\
& -3000000000000 \sin\left(\frac{511}{1000}z\right) \\
& -8790753177z^2 \sin(z) \\
& +809403552846 \sin(z)\cos(z) \\
& -9677616366768 \cos(z) \sin\left(\frac{1}{2}z\right) \\
& -213492529600z^2 \cos(z) \sin\left(\frac{1}{2}z\right) \\
& +200000000000z^2 \cos(z) \sin\left(\frac{511}{1000}z\right) \\
& +213492529600z^2 \cos\left(\frac{1}{2}z\right) \sin(z) \\
& -1600000000000z^2 \cos\left(\frac{1}{2}z\right) \sin\left(\frac{511}{1000}z\right) \\
& -1600000000000z^2 \sin\left(\frac{1}{2}z\right) \cos\left(\frac{511}{1000}z\right) \\
& +200000000000z^2 \sin(z) \cos\left(\frac{511}{1000}z\right) \\
& -3025315653784z^2 \sin\left(\frac{1}{2}z\right) \\
& +2900000000000z^2 \sin\left(\frac{511}{1000}z\right)
\end{aligned}
$$

$$
\begin{aligned}
M_2 = {} & z^2(-545473413128\cos(z) \sin\left(\frac{1}{2}z\right) \\
& + 134900592141\sin(z) \cos(z) \\
& -500000000000 \cos(z) \sin\left(\frac{511}{1000}z\right) \\
& -106746248000 \cos\left(\frac{1}{2}z\right) \sin(z) \\
& +800000000000 \cos\left(\frac{1}{2}z\right) \sin\left(\frac{511}{1000}z\right) \\
& -7454526586872 \sin\left(\frac{1}{2}z\right) \\
& +1932562055859 \sin(z) \\
& +800000000000z^2 \cos\left(\frac{511}{1000}z\right) \sin\left(\frac{1}{2}z\right) \\
& -1000000000000 \sin(z) \cos\left(\frac{511}{1000}z\right) \\
& -7500000000000 \sin\left(\frac{511}{1000}z\right)
\end{aligned}
$$

$$
\begin{aligned}
M_3 = {} & -45000000000000\cos(z) \sin\left(\frac{511}{1000}z\right) \\
& -11595372335154 \sin(z) \\
& + 45000000000000 \sin\left(\frac{511}{1000}z\right) \\
& +11595372335154\sin(z)\cos(z) \\
& -44727159521232\cos(z) \sin\left(\frac{1}{2}z\right) \\
& +8790753177z^2\cos(z)\sin(z) \\
& +36420239384z^2 \cos(z)\sin\left(\frac{1}{2}z\right) \\
& -100000000000z^2\cos(z) \sin\left(\frac{511}{1000}z\right) \\
& -22400000000000z^2 \cos\left(\frac{1}{2}z\right) \sin\left(\frac{511}{1000}z\right) \\
& -22400000000000z^2 \cos\left(\frac{511}{1000}z\right) \sin\left(\frac{1}{2}z\right) \\
& +2800000000000z^2 \cos\left(\frac{511}{1000}z\right) \sin(z)
\end{aligned}
$$

$$
\begin{aligned}
M_4 = {} & z^2(545473413128\cos(z)\sin\left(\tfrac{1}{2}z\right) \\
& + 134900592141\sin(z)\cos(z) \\
& -500000000000\cos(z)\sin\left(\frac{511}{1000}z\right) \\
& -1067462648000\cos\left(\tfrac{1}{2}z\right)\sin(z) \\
& +8000000000000\cos\left(\tfrac{1}{2}z\right)\sin\left(\frac{511}{1000}z\right) \\
& -7454526586872\sin\left(\tfrac{1}{2}z\right) \\
& +1932562055859\sin(z) \\
& +8000000000000\cos\left(\frac{511}{1000}z\right)\sin\left(\tfrac{1}{2}z\right) \\
& -1000000000000\cos\left(\frac{511}{1000}z\right)\sin(z) \\
& -7500000000000\sin\left(\frac{511}{1000}z\right)
\end{aligned}
$$

$$
\begin{aligned}
M_5 = {} & 213492529600z^2\cos(z)\sin\left(\tfrac{1}{2}z\right) \\
& - 200000000000z^2\cos(z)\sin\left(\frac{511}{1000}z\right) \\
& +213492529600z^2\cos\left(\tfrac{1}{2}z\right)\sin(z) \\
& -1600000000000z^2\cos\left(\frac{511}{1000}z\right)\sin\left(\tfrac{1}{2}z\right) \\
& +2952475175016z^2\sin\left(\tfrac{1}{2}z\right) \\
& -8790753177z^2\sin(z) \\
& -1600000000000z^2\cos\left(\frac{511}{1000}z\right)\sin\left(\tfrac{1}{2}z\right) \\
& +200000000000z^2\cos\left(\frac{511}{1000}z\right)\sin(z) \\
& -2700000000000z^2\sin\left(\frac{511}{1000}z\right) \\
& -3131935409232\sin\left(\tfrac{1}{2}z\right) \\
& +3131935409232\cos(z)\sin\left(\tfrac{1}{2}z\right) \\
& -809403552846\sin(z) \\
& +809403552846\sin(z)\cos(z) \\
& +9000000000000\sin\left(\frac{511}{1000}z\right) \\
& -9000000000000\cos(z)\sin\left(\frac{511}{1000}z\right).
\end{aligned}
$$

$$
\begin{aligned}
M_6 = {} & z^2(-545473413128\cos(z)\sin\left(\tfrac{1}{2}z\right) \\
& +134900592141\sin(z)\cos(z) \\
& -500000000000\cos(z)\sin\left(\frac{511}{1000}z\right) \\
& -1067462648000\cos\left(\tfrac{1}{2}z\right)\sin(z) \\
& +8000000000000\cos\left(\tfrac{1}{2}z\right)\sin\left(\frac{511}{1000}z\right) \\
& -7454526586872\sin\left(\tfrac{1}{2}z\right) \\
& +1932562055859\sin(z) \\
& +8000000000000\cos\left(\frac{511}{1000}z\right)\sin\left(\tfrac{1}{2}z\right) \\
& -1000000000000\cos\left(\frac{511}{1000}z\right)\sin(z) \\
& -7500000000000\sin\left(\frac{511}{1000}z\right)\Bigg).
\end{aligned}
$$

$$
\begin{aligned}
M_7 = {} & -30000000000\cos(z)\sin\left(\frac{511}{1000}z\right) \\
& -4002984930\sin(z)+30000000000\sin\left(\frac{511}{1000}z\right) \\
& +1868059634z^2\sin(z) \\
& + 4002984930\sin(z)\cos(z) \\
& +133432831z^2\cos(z)\sin(z) \\
& -1000000000z^2\cos(z)\sin\left(\frac{511}{1000}z\right) \\
& -14000000000\,z^2\sin\left(\frac{511}{1000}z\right),
\end{aligned}
$$

$$
\begin{aligned}
M_8 = {} & z^2(-545473413128\cos(z)\sin\left(\tfrac{1}{2}z\right) \\
& +134900592141\sin(z)\cos(z) \\
& -500000000000\cos(z)\sin\left(\frac{511}{1000}z\right) \\
& -1067462648000\left(\tfrac{1}{2}z\right)\sin(z) \\
& + 8000000000000\cos\left(\tfrac{1}{2}z\right)\sin\left(\frac{511}{1000}z\right) \\
& -7454526586872\sin\left(\tfrac{1}{2}z\right) \\
& +1932562055859\sin(z)
\end{aligned}
$$

$$+8000000000000\cos\left(\frac{511}{1000}z\right)\sin\left(\frac{1}{2}z\right)$$

$$-1000000000000\cos\left(\frac{511}{1000}z\right)\sin(z)$$

$$-7500000000000\sin\left(\frac{511}{1000}z\right)\bigg).$$

$$M_9 = \left(-8\sin\left(\frac{1}{2}z\right)+\sin(z)\right)\left(14z^2-30+z^2\cos(z)+30\cos9z)\right).$$

$$M_{10} = z^2\left(-545473413128\cos(z)\sin\left(\frac{1}{2}z\right)\right.$$

$$+134900592141\sin(z)\cos(z)$$

$$-500000000000\cos(z)\sin\left(\frac{511}{1000}z\right)$$

$$-1067462648000\cos\left(\frac{1}{2}z\right)\sin(z)$$

$$+8000000000000\cos\left(\frac{1}{2}z\right)\sin\left(\frac{511}{1000}z\right)$$

$$-7454526586872\sin\left(\frac{1}{2}z\right)$$

$$+1932562055859\sin(z)$$

$$+8000000000000\cos\left(\frac{511}{1000}z\right)\sin\left(\frac{1}{2}z\right)$$

$$-1000000000000\cos\left(\frac{511}{1000}z\right)\sin(z)$$

$$-7500000000000\sin\left(\frac{511}{1000}z\right)\bigg).$$

$$M_{11} = -4000000000000\sin\left(\frac{511}{1000}z\right)$$

$$+2044000000000\sin(z)$$

$$+42413378651z^2\sin(z),$$

$$M_{12} = 2000000000000\cos\left(\frac{511}{1000}z\right)\sin(z)$$

$$-978000000000\sin(z)$$

$$-2000000000000\cos(z)\sin\left(\frac{511}{1000}z\right)$$

$$+42413378651z^2\cos(z)\sin(z),$$

$$a_{31} = 0,\ a_{32} = -\frac{2(\cos(z)-1)}{z^2},$$

$$a_{41} = -\frac{1}{2}\frac{-2\sin\left(\frac{1}{2}z\right)+\sin(z)}{z^2\sin(z)}$$

$$2\sin\left(\frac{1}{2}z\right)+\sin(z)-3\sin(z)+$$

$$a_{42} = -\frac{2\cos(z)\sin\left(\frac{1}{2}z\right)}{2z^2\sin(z)},$$

$$a_{51} = \frac{1}{14000000000000}\frac{M_{11}}{z^2\sin(z)},$$

$$a_{52} = -\frac{1}{12000000000000}\frac{M_{12}}{z^2\sin(z)}.$$

It is important to note that the original method, that is HM5(5,6, ∞) needs to be recovered as z approaches zero. As, such, Taylor series expansion of the coefficients above is useful. That is

$$b_1 = \frac{89}{5868}+\frac{249211147}{22181040000000}z^4+\ldots$$

$$b_2 = \frac{1355}{3066}+\frac{317510561}{4292400000000}z^4+\ldots$$

$$b_3 = \frac{311}{18132}+\frac{302550317}{76154400000000}z^4+\ldots$$

$$b_4 = \frac{800}{3033}+\frac{743041}{1433092500}z^4+\ldots$$

$$b_5 = \frac{100000000000}{381720407859}+\frac{1963647500}{72145157085351}z^4+\ldots,$$

$$a_{42} = \frac{5}{16}+\frac{1}{768}z^2+\frac{13}{18432}z^4+\frac{1381}{20643840}z^6+\ldots,$$

$$a_{43} = 0,\ a_{31} = 0,$$

$$a_{32} = 1-\frac{1}{12}z^2+\frac{1}{360}z^4+\frac{1}{20160}z^6+\ldots,$$

$$a_{41} = \frac{1}{16}+\frac{5}{768}z^2+\frac{61}{92160}z^4+\frac{277}{4128768}z^6+\ldots,$$

$$a_{51} = -\frac{209298067349}{4000000000000}-\frac{782399344263551}{12000000000000000}z^2+\ldots,$$

$$a_{52} = -\frac{166436655651}{2000000000000}+\frac{493208130588551}{120000000000000000}z^2+\ldots,$$

It is clear that as z approaches zero the original method HM5(5,6) is recovered.

Next is to verify the algebraic order of the method. To check if the method is order five as claimed, we substitute the coefficients of the method into the algebraic order conditions up to order five and take the Taylor series expansion of each.

Order 1:

$$\sum b_i = 1$$

Order 2:

$$\sum b_i c_i = \frac{11}{1200000}$$

Order3:

$$\sum b_i c_i^2 = \frac{1}{6} - \frac{1131959}{75600000000} z^4 + \ldots,$$

$$\sum b_i a_{ij} = \frac{1}{12} - \frac{11}{60480} z^4,$$

Order 4:

$$\sum b_i c_i^3 = 0$$

$$\sum b_i c_i a_{ij} = \frac{1}{12} - \frac{11}{144000} z^4 + \ldots,$$

$$\sum b_i a_{ij} c_j = -\frac{11}{1200000} z^2,$$

Order5:

$$\sum b_i c_i^4 = \frac{1}{15}$$

$$\sum b_i c_i^2 a_{ij} = \frac{1}{30} - \frac{51793}{48000000} z^2 + \ldots,$$

$$\sum b_i c_i a_{ij} c_j = -\frac{1}{60} - \frac{62331370}{360000000} z^2 + \ldots,$$

$$\sum b_i a_{ij} a_{ik} = \frac{7}{120} - \frac{166379}{144000000} z^2$$

$$\sum b_i a_{ij} c_j^2 = \frac{1}{180} - \frac{11}{1200000} z^2 + \ldots$$

$$\sum b_i a_{ij} a_{jk} c_j = \frac{1}{360} - \frac{1}{4320} z^2 + \ldots$$

It can be seen from above that as z tends to zero the algebraic order five conditions are recovered, which implies that the coefficients of trigonometrically fitted fifth order method satisfies algebraic order five conditions.

RESULTS

Presented in this section are numerical results of the trigonometrically fitted zero-dissipative fifth order hybrid method derived in the paper denoted by TFZDHM5. The method is applied to Problems 1-6 presented as follows:

Problem 1 (Non-linear problem)

$$y''(x) = -\pi \sin(\pi x), \; y(0) = 0, y'(0) = 1.$$

Exact solution: $y(x) = \lambda = \pi$.
Source: Dizicheh et al. 2012

Problem 2 (Homogeneous problem)

$$y''(x) = -64y(x), y(0) = 1, y'(0) = -2,$$

Exact solution is $y(x) = -\frac{1}{4}\sin(8x) + \cos(8x)$, $\lambda = 8$
Source: Senu et al. 2009

Problem 3 (Homogeneous problem)

$y''(x) = -100y(x)$, $y(0) = 1$, $y'(0) = -2$ and the fitted frequency, $\lambda = 10$

Exact solution is $y(x) = -\frac{1}{5}\sin(10x) + \cos(10x)$
Source: Senu et al. 2009

Problem 4 (Two body problem)

$$y_1''(x) = -\frac{y_1}{\left(y_1^2 + y_2^2\right)^{3/2}}, y_1(0) = 1, y_1'(0) = 0$$

$$y_2''(x) = -\frac{y_2}{\left(y_1^2 + y_2^2\right)^{3/2}}, y_2(0) = 0, y_2'(0) = 1$$

Exact solution: $y_1(x) = \cos(x)$, $y_2(x) = \sin(x)$
$\lambda = 1$
Source: Senu et al. 2009

Problem 5 (Inhomogeneous problem)

$$y''(x) = -y(x) + x, y(0) = 1, y'(0) = 2,$$

Exact solution: $y(x) = \sin(x) + \cos(x) + x$
$\lambda = 1$
Source: Al-Khasawneh et al. 2007

Problem 6 (Duffing problem)

μ_5 is 0.304014×10^{-6},
μ_7 is 0.374×10^{-9},
$\mu_9 < 10^{-12}$;
and $\lambda = 1$.

Exact solution:

$y(x) = \sum_{i=0}^{4} \mu_{2i+1} \cos[(2i+1)ux]$,
where μ_1 is 0.200179477536; μ_3 is $0.246946143 \times 10^{-3}$, μ_5 is 0.304014×10^{-6}.
Source: Van de Vyver 2005

The results are tabulated together with those of existing codes in the literature for the purpose of comparison. The interval of integration are taken to be 100, 1000 and 10000, respectively.

Numerical results of trigonometrically fitted zero-dissipative fifth order hybrid method applied to Problems 1-6 with different step sizes and integration intervals are tabulated in this section. Both small and large integration intervals are considered to measure the stability of the methods when solving highly oscillatory problems.

It can be seen from Tables 2-7 that the accuracy of TFZDHM5 diminishes as h grows smaller, especially for problems with small frequencies. This is because the

TABLE 2. Accuracy Comparison of TFZDHM5 for problem 1

h	Method	100	1000	10000
1/2	TFZDFHM5	4.300200×10^{-46}	2.317910×10^{-45}	1.545223×10^{-41}
	TFHM5	$1.286527 \times 10^{+02}$	$1.310157 \times 10^{+04}$	$1.312520 \times 10^{+06}$
	HM5(5,6,	1.427249×10^{-02}	1.427240×10^{-01}	$1.427249 \times 10^{+01}$
	MKRN5(8,7)	3.794000×10^{-04}	3.234150×10^{-03}	3.234153×10^{-02}
1/4	TFZDFHM5	4.300200×10^{-46}	2.317910×10^{-45}	1.039012×10^{-41}
	TFHM5	2.184617×10^{-01}	$2.184617 \times 10^{+00}$	$2.184617 \times 10^{+01}$
	HM5(5,6,	2.676400×10^{-04}	2.676430×10^{-03}	4.270000×10^{-02}
	MKRN5(8,7)	1.085000×10^{-05}	4.751000×10^{-05}	4.751000×10^{-06}
1/6	TFZDFHM5	3.240000×10^{-47}	3.263000×10^{-46}	1.420675×10^{-41}
	TFHM5	1.731204×10^{-02}	1.731203×10^{-01}	$1.731203 \times 10^{+00}$
	HM5(5,6,	4.360000×10^{-06}	4.364770×10^{-05}	4.364800×10^{-04}
	MKRN5(8,7)	3.320552×10^{-07}	7.312912×10^{-07}	7.310000×10^{-06}

TFZDHM5: Trigonometrically fitted zero dissipative fifth order hybrid method derived in this paper
HM5(5,6,): zero dissipative fifth order five stage sixth order dispersive hybrid method derived in Yusouf Dauda (2014)
THFM: Trigonometrically fitted fifth order hybrid method derived in Dizicheh et al. (2012)
MRKN 5(8,7): Fifth order explicit Rung-Kutta Nystrom method for periodic IVPs in Mohamad et al. (2012)

TABLE 3. Accuracy Comparison of TFZDHM5 for problem 2

h	Method	100	1000	10000
2	TFZDFHM5	4.300200×10^{-46}	2.317910×10^{-45}	1.545223×10^{-45}
	TFHM5	$1.516560 \times 10^{+21}$	$1.990486 \times 10^{+21}$	$3.019643 \times 10^{+21}$
	HM5(5,6,	$1.752494 \times 10^{+22}$	$7.501300 \times 10^{+23}$	$1.548587 \times 10^{+23}$
	MKRN5(8,7)	$6.785154 \times 10^{+29}$	$3.991504 \times 10^{+29}$	$1.981094 \times 10^{+29}$
1	TFZDFHM5	1.162215×10^{-46}	4.317910×10^{-44}	1.545223×10^{-45}
	TFHM5	$3.574758 \times 10^{+24}$	$3.702791 \times 10^{+24}$	$3.019643 \times 10^{+21}$
	HM5(5,6,	$3.754850 \times 10^{+26}$	$2.876880 \times 10^{+26}$	$1.548587 \times 10^{+23}$
	MKRN5(8,7)	$1.445355 \times 10^{+29}$	$4.134780 \times 10^{+29}$	$1.981094 \times 10^{+29}$
1/2	TFZDFHM5	3.295900×10^{-46}	3.263000×10^{-46}	1.420675×10^{-45}
	TFHM5	$8.114588 \times 10^{+37}$	$1.077701 \times 10^{+37}$	$1.538857 \times 10^{+37}$
	HM5(5,6,	$5.151363 \times 10^{+11}$	$1.491524 \times 10^{+11}$	$6.176056 \times 10^{+11}$
	MKRN5(8,7)	$4.475182 \times 10^{+92}$	$1.650440 \times 10^{+92}$	$7.682290 \times 10^{+92}$

TFZDHM5: Trigonometrically fitted zero dissipative fifth order hybrid method derived in this paper
HM5(5,6,): zero dissipative fifth order five stage sixth order dispersive hybrid method derived in Yusouf Dauda (2014)
THFM: Trigonometrically fitted fifth order hybrid method derived in Dizicheh et al. (2012)
MRKN 5(8,7): Fifth order explicit Rung-Kutta Nystrom method for periodic IVPs in Mohamad et al. (2012)

TABLE 4. Accuracy Comparison of TFZDHM5 for problem 3

h	Method	100	1000	10000
2	TFZDFHM5	3.566400×10^{-46}	3.952690×10^{-45}	4.199172×10^{-44}
	TFHM5	$9.854612 \times 10^{+24}$	$4.096083 \times 10^{+24}$	$6.304650 \times 10^{+24}$
	HM5(5,6,	$9.761757 \times 10^{+25}$	$4.683804 \times 10^{+26}$	$3.029045 \times 10^{+26}$
	MKRN5(8,7)	$4.094894 \times 10^{+33}$	$6.232955 \times 10^{+33}$	$4.161089 \times 10^{+33}$
1	TFZDFHM5	4.309600×10^{-46}	4.577710×10^{-45}	4.625808×10^{-44}
	TFHM5	$4.854673 \times 10^{+31}$	$1.671032 \times 10^{+31}$	$3.901532 \times 10^{+31}$
	HM5(5,6,	$3.698753 \times 10^{+32}$	$5.048806 \times 10^{+33}$	$1.133764 \times 10^{+33}$
	MKRN5(8,7)	$1.891302 \times 10^{+38}$	$1.088873 \times 10^{+38}$	$4.356526 \times 10^{+38}$
1/2	TFZDFHM5	5.549000×10^{-47}	6.006100×10^{-46}	6.191350×10^{-45}
	TFHM5	$4.691006 \times 10^{+17}$	$9.412910 \times 10^{+17}$	$9.960990 \times 10^{+17}$
	HM5(5,6,	$2.206692 \times 10^{+22}$	$1.479780 \times 10^{+23}$	$2.721132 \times 10^{+23}$
	MKRN5(8,7)	$3.241873 \times 10^{+23}$	$1.244249 \times 10^{+23}$	$8.630213 \times 10^{+23}$

TFZDHM5: Trigonometrically fitted zero dissipative fifth order hybrid method derived in this paper
HM5(5,6,): zero dissipative fifth order five stage sixth order dispersive hybrid method derived in Yusouf Dauda (2014)
THFM: Trigonometrically fitted fifth order hybrid method derived in Dizicheh et al. (2012)
MRKN 5(8,7): Fifth order explicit Rung-Kutta Nystrom method for periodic IVPs in Mohamad et al. (2012)

TABLE 5. Accuracy Comparison of TFZDHM5 for problem 4

h	Method	100	1000	10000
2	TFZDFHM5	1.042171×10^{-44}	8.443420×10^{-43}	5.816611×10^{-41}
	TFHM5	1.700000×10^{-06}	1.790700×10^{-04}	2.409809×10^{-02}
	HM5(5,6,	1.229167×10^{-01}	$2.012403 \times 10^{+00}$	$2.092650 \times 10^{+00}$
	MKRN5(8,7)	2.159574×10^{-02}	$1.619361 \times 10^{+00}$	$1.996332 \times 10^{+00}$
1	TFZDFHM5	1.640399×10^{-43}	1.798316×10^{-41}	1.813261×10^{-39}
	TFHM5	1.700000×10^{-06}	1.790700×10^{-04}	2.409809×10^{-04}
	HM5(5,6,	1.229160×10^{-01}	$2.012403 \times 10^{+00}$	$2.092650 \times 10^{+00}$
	MKRN5(8,7)	2.159574×10^{-02}	$1.619361 \times 10^{+00}$	$1.996332 \times 10^{+00}$
1/2	TFZDFHM5	2.319447×10^{-46}	1.878809×10^{-40}	1.835145×10^{-38}
	TFHM5	2.254001×10^{-10}	2.309894×10^{-08}	2.340000×10^{-06}
	HM5(5,6,	1.251650×10^{-03}	1.142914×10^{-01}	$1.999180 \times 10^{+00}$
	MKRN5(8,7)	1.992300×10^{-04}	1.524123×10^{-02}	$1.343535 \times 10^{+00}$

TFZDHM5: Trigonometrically fitted zero dissipative fifth order hybrid method derived in this paper
HM5(5,6,): zero dissipative fifth order five stage sixth order dispersive hybrid method derived in Yusouf Dauda (2014)
THFM: Trigonometrically fitted fifth order hybrid method derived in Dizicheh et al. (2012)
MRKN 5(8,7): Fifth order explicit Rung-Kutta Nystrom method for periodic IVPs in Mohamad et al. (2012)

TABLE 6. Accuracy Comparison of TFZDHM5 for problem 5

h	Method	100	1000	10000
2	TFZDFHM5	1.400000×10^{-47}	7.200000×10^{-46}	5.060000×10^{-44}
	TFHM5	1.122416×10^{-02}	1.177540×10^{-01}	9.108614×10^{-01}
	HM5(5,6,	4.277947×10^{-01}	$2.826494 \times 10^{+00}$	$2.834484 \times 10^{+00}$
	MKRN5(8,7)	3.127991×10^{-01}	$1.287182 \times 10^{+00}$	$1.415626 \times 10^{+00}$
1	TFZDFHM5	2.100000×10^{-47}	1.060000×10^{-45}	2.010000×10^{-44}
	TFHM5	8.608921×10^{-07}	8.860000×10^{-06}	8.916000×10^{-05}
	HM5(5,6,	3.982780×10^{-03}	4.085921×10^{-02}	4.095443×10^{-01}
	MKRN5(8,7)	1.567380×10^{-03}	7.560990×10^{-03}	7.209059×10^{-02}
1/2	TFZDFHM5	7.310000×10^{-46}	8.070000×10^{-45}	7.250000×10^{-44}
	TFHM5	1.552306×10^{-10}	1.570140×10^{-09}	1.576863×10^{-08}
	HM5(5,6,	5.525000×10^{-05}	5.650200×10^{-04}	5.692420×10^{-03}
	MKRN5(8,7)	3.563000×10^{-05}	3.642000×10^{-05}	2.516400×10^{-04}

TFZDHM5: Trigonometrically fitted zero dissipative fifth order hybrid method derived in this paper
HM5(5,6,): zero dissipative fifth order five stage sixth order dispersive hybrid method derived in Yusouf Dauda (2014)
THFM: Trigonometrically fitted fifth order hybrid method derived in Dizicheh et al. (2012)
MRKN 5(8,7): Fifth order explicit Rung-Kutta Nystrom method for periodic IVPs in Mohamad et al. (2012)

TABLE 7. Accuracy Comparison of TFZDHM5 for problem 6

h	Method	100	1000	10000
2	TFZDFHM5	1.369505×10^{-02}	2.723286×10^{-02}	2.723286×10^{-02}
	TFHM5	8.361090×10^{-03}	1.061014×10^{-02}	1.061014×10^{-02}
	HM5(5,6,	1.221149×10^{-02}	2.665768×10^{-02}	2.665768×10^{-02}
	MKRN5(8,7)	3.747842×10^{-02}	6.096144×10^{-02}	1.268348×10^{-01}
1	TFZDFHM5	2.931900×10^{-04}	4.870000×10^{-04}	4.870000×10^{-04}
	TFHM5	2.147100×10^{-04}	2.586400×10^{-04}	2.590600×10^{-04}
	HM5(5,6,	3.768900×10^{-04}	1.137130×10^{-03}	1.137130×10^{-03}
	MKRN5(8,7)	1.040600×10^{-04}	3.540600×10^{-04}	3.212700×10^{-04}
1/2	TFZDFHM5	4.420000×10^{-06}	6.060000×10^{-06}	6.060000×10^{-06}
	TFHM5	2.500000×10^{-06}	8.160000×10^{-09}	8.160000×10^{-09}
	HM5(5,6,	2.500000×10^{-06}	7.980000×10^{-06}	7.980000×10^{-06}
	MKRN5(8,7)	1.780000×10^{-06}	3.170000×10^{-06}	8.260000×10^{-06}

TFZDHM5: Trigonometrically fitted zero dissipative fifth order hybrid method derived in this paper
HM5(5,6,): zero dissipative fifth order five stage sixth order dispersive hybrid method derived in Yusouf Dauda (2014)
THFM: Trigonometrically fitted fifth order hybrid method derived in Dizicheh et al. (2012)
MRKN 5(8,7): Fifth order explicit Rung-Kutta Nystrom method for periodic IVPs in Mohamad et al. (2012)

method approaches the original method as $h \rightarrow 0$. For problems with large frequency, TFZDHM5 performs far better than the original method as well as other methods considered.

CONCLUSION

In this paper trigonometrically fitted hybrid method based on the existing zero-dissipative hybrid method was derived. The method was applied to highly oscillatory problems. The results obtained were compared with those of existing methods in the literature. From the numerical results obtained, we can conclude that the new trigonometrically fitted hybrid methods approximate the solution of highly oscillatory problems better than trigonometrically fitted hybrid methods that are based on higher dispersive and dissipative methods.

Digits=50 was used in *MAPLE* 16 environment for implementing all the methods on the problems.

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