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# A Study on the $S^2$-EWMA Chart for Monitoring the Process Variance based on the MRL Performance

(Suatu Kajian Carta $S^2$-EWMA bagi Memantau Varians Proses Berdasarkan Prestasi MRL)

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ABSTRACT

*The existing optimal design of the fixed sampling interval $S^2$-EWMA control chart to monitor the sample variance of a process is based on the average run length (ARL) criterion. Since the shape of the run length distribution changes with the magnitude of the shift in the variance, the median run length (MRL) gives a more meaningful explanation about the in-control and out-of-control performances of a control chart. This paper proposes the optimal design of the $S^2$-EWMA chart, based on the MRL. The Markov chain technique is employed to compute the MRLs. The performances of the $S^2$-EWMA chart, double sampling (DS) $S^2$ chart and S chart are evaluated and compared. The MRL results indicated that the $S^2$-EWMA chart gives better performance for detecting small and moderate variance shifts, while maintaining almost the same sensitivity as the DS $S^2$ and S charts toward large variance shifts, especially when the sample size increases.*

*Keywords: Exponentially weighted moving average (EWMA); Markov chain; median run length (MRL); sample variance*

ABSTRAK

*Reka bentuk optimum carta kawalan EWMA-$S^2$ selang pensampelan tetap yang digunakan untuk memantau proses sampel varians adalah berdasarkan kriteria panjang larian purata (ARL). Oleh sebab bentuk taburan panjang larian berubah dengan magnitud anjakan dalam varians, maka panjang larian median (MRL) memberi penjelasan yang lebih bermakna tentang prestasi terkawal dan luar kawalan carta kawalan. Kertas kerja ini mencadangkan reka bentuk optimum untuk carta EWMA-$S^2$ berdasarkan MRL. Teknik rantai Markov digunakan untuk mengira MRL. Prestasi carta-carta EWMA-$S^2$, DS $S^2$ dan S telah dinilai dan dibandingkan. Keputusan MRL menunjukkan bahawa carta EWMA-$S^2$ memberikan prestasi yang lebih baik untuk mengesan anjakan varians yang kecil dan sederhana di samping mengekalkan kepekaan yang hampir sama dengan carta-carta DS $S^2$ dan S terhadap anjakan varians yang besar, terutamanya apabila saiz sampel meningkat.*

*Kata kunci: Panjang larian median; purata bergerak berpemberat eksponen (EWMA); rantai Markov; varians sampel*

## INTRODUCTION

Control charts are the core tools in the application of statistical process control (SPC) to determine whether a process is in statistical control. As different processes require different methods of monitoring, different kinds of control charts have been developed by researchers. Roberts (1959) was the first person to introduce the exponentially weighted moving average (EWMA) control chart and since then, the EWMA control chart has been well accepted and widely used by practitioners. The EWMA chart is good for detecting small process shifts (Razmy & Peiris 2013).

To date, there are many extensions on the EWMA chart and the more important ones are briefly discussed as follows:

In order to improve the properties and design strategies of the EWMA chart for the process mean, Simões et al. (2010) optimized the designs of the EWMA chart with a variable smoothing constant (AEWMA) with regards to pairs of shifts in the process mean. In the same year, Li et al. (2010) introduced the nonparametric EWMA chart for detecting mean shifts. A new nonparametric EWMA sign control chart was proposed by Yang et al. (2011) for monitoring and detecting possible deviations from the process target. In addition, a nonparametric EWMA signed-rank chart was developed by Graham et al. (2011) for monitoring the process location.

The number of defective units increase with the increase of the process variance as, it is crucial to monitor changes in the process variance. Thus, a lot of effort has been put in to design EWMA charts for monitoring the process dispersion. Chang and Gan (1994) designed the one-sided optimal EWMA chart to monitor process variance. Castagliola (2005) proposed the fixed sample size and sampling interval (FSSI) $S^2$-EWMA control chart to monitor the sample variance of a process. Later on, an extension on the FSSI $S^2$-EWMA chart, i.e. the variable sampling interval (VSI) $S^2$-EWMA chart was developed by Castagliola et al. (2007). Castagliola et al. (2008) discussed the construction of a variable sample size (VSS) version of the static FSSI $S^2$-EWMA chart to monitor the stability of the process dispersion. Eyvazian et al. (2008) proposed an exponentially weighted moving sample variance chart to monitor process variance when the sample size is one. Shu (2008) extended the adaptive EWMA chart for process location to monitor the process dispersion. Razmy and Peiris (2013) designed the EWMA chart for monitoring standardized process variance.

The performance of control charts for monitoring a process in most previous studies is usually measured using the average run length (ARL) because of the following reasons: The derivation of the run length distribution is particularly hard in most cases and the in-control run length distribution is approximately geometric, therefore it can be approximately characterized by the ARL (Gan 1992). The ARL is defined as the average (expected) number of sample points that must be plotted on the chart before the first out-of-control signal is detected (Montgomery 2009). In other words, ARL is a measure of the speed of a chart in detecting the occurrence of assignable causes.

However, interpretation based on the ARL can be misleading (Gan 1993a) as the in-control run length distribution of a EWMA chart is highly skewed. Furthermore, the shape of the run length distribution changes with the magnitude of the shift in the variance. This fact is further supported by the findings in Teoh and Khoo (2012) who reported on the skewness of the run length distribution changes with the size of the process mean shifts. Therefore, the median run length (MRL) actually gives a more meaningful explanation about the in-control and out-of-control performances of a control chart compared to the ARL (Gan 1994, 1993a). For a run length distribution which changes from a highly skewed distribution when the shift is small to an almost symmetric distribution when the shift is large, the MRL is more readily understood by practitioners. In contrast, interpretation based on the ARL could be misleading.

The MRL is defined as the median number of sample points that must be plotted on the chart before the first out-of-control signal is issued. In other words, the MRL is the 50$^{th}$ percentage point of the run length distribution. Chakraborti (2007), Gan (1993a), Radson and Boyd (2005) and Thaga (2003) to name a few, have all criticized the use of ARL as a sole measure of the performance of a chart as it is insufficient. Furthermore, Di Bucchianico et al. (2005) also commented that when the run length distribution is highly skewed, it is less meaningful to judge the performance of a control chart by considering its ARL only.

The FSSI $S^2$-EWMA chart proposed by Castagliola (2005) is optimally designed based on the ARL. Gan (1994) noted that a better understanding of a control chart via the use of MRL helps to increase the confidence of quality control practitioners and engineers. The main contributions of this work are to present a procedure to optimally design the FSSI $S^2$-EWMA chart of Castagliola (2005), using the MRL criterion as described in Gan (1994, 1993a & 1993b) and to develop a SAS program to compute the optimal parameters of the chart.

The layout of this paper is as follows: The next section introduces the FSSI $S^2$-EWMA chart and followed by the optimal design of the chart based on MRL is presented in the section that follows. Next is the study and comparison of the MRL performances of the $S^2$-EWMA, double sampling (DS) $S^2$ and $S$ charts. Conclusions and suggestions for future works are drawn in last section. The Markov chain approach employed to compute the MRL of the $S^2$-EWMA chart is discussed in the Appendix.

## THE $S^2$-EWMA CONTROL CHART

Let $X_{k,1}, X_{k,2}, \ldots, X_{k,n}$ be a sample of $n$ independent random variables, having a normal $N(\mu, \sigma_0^2)$ distribution, where $\mu$ is the process mean, $\sigma_0$ is the nominal process standard deviation and $k$ is the sample number. As the $S^2$-EWMA chart is used to monitor the process dispersion, an out-of-control occurs when the standard deviation shifts from $\sigma_0$ to $\sigma_1$, where the magnitude of this shift is measured through the parameter $\tau = \frac{\sigma_1}{\sigma_0}$, while the mean remains at its nominal value $\mu$. In this paper, $\sigma_0$ is assumed to be known. Let $S_k^2$ be the variance of sample $k$, i.e.

$$S_k^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{k,j} - \bar{X}_k\right)^2, \tag{1}$$

where $\bar{X}_k$ is the mean of sample $k$. In order to monitor the process variance, Castagliola (2005) suggested to apply the following transformation on $S_k^2$, i.e.

$$T_k = a + b\ln(S_k^2 + c), \tag{2}$$

where $a$, $b$ and $c > 0$ (in order to avoid problems with the logarithmic transformation) are three constants and then, to use the classical EWMA approach on the $T_k$ statistic, i.e.

$$Z_k = (1-\lambda)Z_{k-1} + \lambda T_k, \tag{3}$$

where $\lambda$ is a smoothing constant satisfying $0 < \lambda \leq 1$. The main motivation of this method is that if the constants $a$, $b$ and $c$ are judiciously selected, then the distribution of $T_k$ will be quasi-symmetrical and will look like a standard normal distribution. The control limits of the $S^2$-EWMA control chart (corresponding to the $Z_k$ statistic) are (Castagliola 2005)

$$LCL = E(T_k) - K \times \sqrt{\frac{\lambda}{2-\lambda}} \times \sigma(T_k), \tag{4}$$

and

$$UCL = E(T_k) + K \times \sqrt{\frac{\lambda}{2-\lambda}} \times \sigma(T_k), \tag{5}$$

where $K$ is a positive constant, $E(T_k)$ and $\sigma(T_k)$ are the theoretical mean and standard deviation of $T_k$. The constants $a$, $b$ and $c$ are equal to (Castagliola 2005)

$$b = B(n), \tag{6}$$

$$c = C(n)\sigma_0^2, \tag{7}$$

and

$$a = A(n) - 2B(n)ln(\sigma_0), \tag{8}$$

where $A(n)$, $B(n)$ and $C(n)$ are three functions depending only on the sample size $n$. The closed forms of these functions are shown in Castagliola (2005). The probability density function (pdf) $f_{T_k}(t|n)$ of $T_k$ whose distribution depends only on $n$, derived by Castagliola (2005) is

$$f_{T_k}(t|n) = \frac{1}{B(n)} exp\left(\frac{t - A(n)}{B(n)}\right) f_G\left\{exp\left(\frac{t - A(n)}{B(n)}\right) - C(n)\middle| \frac{n-1}{2}, \frac{2}{n-1}\right\}, \tag{9}$$

where $f_G$ is the pdf of a gamma distribution with parameters $\frac{n-1}{2}$ and $\frac{2}{n-1}$. This $f_{T_k}(t|n)$ pdf is important since it allows the calculation of the values of $E(T_k)$ and $\sigma(T_k)$ independently of the value of $\sigma_0$. The computation of $E(T_k)$ and $\sigma(T_k)$ was obtained by Castagliola (2005) via numerical quadrature. Note that the values of $E(T_k)$ are very close to zero. In fact, these values are so close to zero that assuming $E(T_k) = 0$ is a very good approximation. Castagliola (2005) has also shown that a reasonable value of $Z_0$ can be obtained through

$$Z_0 = A(n) + B(n)ln[1 + C(n)]. \tag{10}$$

As it can be noticed, $Z_0$ depends only on $n$ and not on $\sigma_0$. Note that the value of $Z_0$ is also close to zero and it can be replaced by zero in practice with little practical effect.

Castagliola (2005) showed that the derivative of $T_k$ has the distribution of the transformed random variable $\tau^2 S^2$ with pdf

$$f_{T'_k}(t|n,\tau) = \frac{1}{B(n)} exp\left(\frac{t - A(n)}{B(n)}\right) f_G\left\{exp\left(\frac{t - A(n)}{B(n)}\right) - C(n)\middle| \frac{n-1}{2}, \frac{2\tau^2}{n-1}\right\}. \tag{11}$$

For this reason, the distribution $f_{T'_k}(t|n,\tau)$ of $T'_k$ depends only on $n$ and $\tau$.

## OPTIMAL DESIGN OF THE $S^2$-EWMA CHART

The optimal parameters of the $S^2$-EWMA chart are computed using the Markov chain approach presented in the Appendix. A chart is optimal in detecting a shift if it yields the smallest possible out-of-control MRL ($MRL_1$), for a specified value of the shift in the process variance, $\tau = \frac{\sigma_1}{\sigma_0}$. More than one optimal parameter combination may exist, for a shift $\tau$ because the MRL is a discrete integer. For this situation, the $(\lambda, K)$ combination corresponding to the smallest $\lambda$, of all optimal $\lambda$'s in the range $[a, b]$, where $0.050 \le a < b \le 1$, is chosen as the optimal parameter combination.

The following steps are recommended in an optimal design of the $S^2$-EWMA chart for detecting shifts in the process variance:

Step 1. Choose the desired in-control MRL ($MRL_0$) value and the sample size, $n$. For an equal footing comparison with Castagliola's (2005) study, $MRL_0 = 370$ (corresponding to the classical $\pm 3\sigma$ limits for a control chart) and $MRL_0 = 200$ (also considered by Crowder & Hamilton 1992), while $n$ = 3, 5, 7 and 9 are considered.

Step 2. Initialize $\lambda$ = 0.050. Note that smaller values of $\lambda$ (i.e. $\lambda < 0.050$) causes numerical difficulty in evaluating the MRLs. This setback was also pointed out by Crowder and Hamilton (1992) for the ARL case.

Step 3. Decide on the desired magnitude of a shift in the process variance, denoted by $\tau$, for which a quick detection is required.

Step 4. When the process is in-control and operates at the nominal variance (i.e. $\sigma_1 = \sigma_0$) or equivalently $\tau$ = 1, determine the value of $K$, in computing LCL and $UCL$ in (4) and (5), respectively, so that the $MRL_0$ value in Step 1 is satisfied, for a particular combination of $(\lambda, K)$. Repeat the process of finding suitable values of $K$ to attain the desired $MRL_0$, for the $\lambda$ values of 0.051, 0.052, ..., 1. Thus, there are 951 $(\lambda, K)$ combinations considered for the $S^2$-EWMA chart.

Step 5. Compute the $MRL_1$ values for all the combinations of $(\lambda, K)$ in Step 4, based on the $\tau$ value specified in Step 3.

Step 6. Identify the $(\lambda^*, K^*)$ combination having the lowest $MRL_1$ value as the optimal parameter combination. Then the optimal $(\lambda^*, K^*)$ combination satisfies the constraints in (12) and (13).

$$\mathrm{MRL}(1, \lambda^*, K^*, n, t) = \mathrm{MRL}_0. \tag{12}$$

$$\mathrm{MRL}(\tau, \lambda^*, K^*, n, t) = \min_{\lambda, K} \mathrm{MRL}(\tau, \lambda, K, n, t). \tag{13}$$

A program is written in the Statistical Analysis Software (SAS) version 9.1.3, incorporating the above 6-steps procedure to compute the optimal $(\lambda^*, K^*)$ combination. The program is available upon request from the first author. Tables 1 and 2 in the following section present the computed optimal $(\lambda^*, K^*)$ combinations for the $S^2$-EWMA chart, for $n \in \{3, 5, 7, 9\}$ and $ARL_0 \in \{200, 370\}$. The optimal parameters for the $S^2$-EWMA chart are obtained via the Markov chain approach.

## MRL PERFORMANCE COMPARISON

Tables 1 and 2 provide the optimal $(\lambda^*, K^*)$ combinations and the corresponding minimum $MRL_1$s, for process variance shift $\tau \in (0.5, 2)$ and $n \in \{3, 5, 7, 9\}$. Table 1 corresponds to $MRL_0 = 370$ while Table 2 corresponds to

TABLE 1. $S^2$-EWMA Chart - Optimal ($\lambda^*$, $K^*$) combinations and the corresponding minimum $MRL_1$s, for $n$ = 3, 5, 7, 9 and $MRL_0$ = 370

| $\tau$ | $n$ = 3 | | | $n$ = 5 | | | $n$ = 7 | | | $n$ = 9 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\lambda^*$ | $K^*$ | MRL* | $\lambda^*$ | $K^*$ | MRL* | $\lambda^*$ | $K^*$ | MRL* | $\lambda^*$ | $K^*$ | MRL* |
| 0.5 | 0.090 | 2.807 | 13 | 0.157 | 3.041 | 6 | 0.201 | 3.135 | 4 | 0.248 | 3.191 | 3 |
| 0.6 | 0.081 | 2.786 | 18 | 0.110 | 2.985 | 9 | 0.142 | 3.090 | 6 | 0.339 | 3.202 | 4 |
| 0.7 | 0.071 | 2.760 | 28 | 0.088 | 2.934 | 14 | 0.147 | 3.095 | 9 | 0.155 | 3.136 | 7 |
| 0.8 | 0.050 | 2.697 | 53 | 0.081 | 2.920 | 25 | 0.076 | 2.959 | 18 | 0.087 | 3.019 | 14 |
| 0.9 | 0.050 | 2.697 | 151 | 0.050 | 2.799 | 68 | 0.050 | 2.844 | 48 | 0.050 | 2.864 | 39 |
| 1.1 | 0.050 | 2.697 | 49 | 0.050 | 2.799 | 42 | 0.050 | 2.844 | 35 | 0.050 | 2.864 | 30 |
| 1.2 | 0.050 | 2.697 | 16 | 0.050 | 2.799 | 15 | 0.050 | 2.844 | 13 | 0.050 | 2.864 | 11 |
| 1.3 | 0.050 | 2.697 | 9 | 0.050 | 2.799 | 9 | 0.050 | 2.844 | 8 | 0.273 | 3.196 | 6 |
| 1.4 | 0.050 | 2.697 | 7 | 0.050 | 2.799 | 6 | 0.143 | 3.091 | 5 | 0.214 | 3.178 | 4 |
| 1.5 | 0.050 | 2.697 | 5 | 0.050 | 2.799 | 5 | 0.092 | 3.005 | 4 | 0.210 | 3.176 | 3 |
| 1.6 | 0.050 | 2.697 | 4 | 0.050 | 2.799 | 4 | 0.150 | 3.099 | 3 | 0.430 | 3.195 | 2 |
| 1.7 | 0.050 | 2.697 | 4 | 0.175 | 3.054 | 3 | 0.387 | 3.152 | 2 | 0.212 | 3.177 | 2 |
| 1.8 | 0.050 | 2.697 | 3 | 0.050 | 2.799 | 3 | 0.214 | 3.141 | 2 | 0.118 | 3.087 | 2 |
| 1.9 | 0.050 | 2.697 | 3 | 0.280 | 3.086 | 2 | 0.126 | 3.070 | 2 | 0.626 | 3.155 | 1 |
| 2.0 | 0.050 | 2.697 | 3 | 0.177 | 3.055 | 2 | 0.745 | 3.085 | 1 | 0.455 | 3.191 | 1 |

TABLE 2. $S^2$-EWMA Chart - Optimal ($\lambda^*$, $K^*$) combinations and the corresponding minimum $MRL_1$s, for $n$ = 3, 5, 7, 9 and $MRL_0$ = 200

| $\tau$ | $n$ = 3 | | | $n$ = 5 | | | $n$ = 7 | | | $n$ = 9 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\lambda^*$ | $K^*$ | MRL* | $\lambda^*$ | $K^*$ | MRL* | $\lambda^*$ | $K^*$ | MRL* | $\lambda^*$ | $K^*$ | MRL* |
| 0.5 | 0.110 | 2.603 | 12 | 0.215 | 2.865 | 6 | 0.156 | 2.881 | 4 | 0.189 | 2.945 | 3 |
| 0.6 | 0.116 | 2.632 | 15 | 0.115 | 2.763 | 8 | 0.189 | 2.913 | 5 | 0.190 | 2.946 | 4 |
| 0.7 | 0.076 | 2.552 | 24 | 0.103 | 2.738 | 12 | 0.144 | 2.865 | 8 | 0.197 | 2.951 | 6 |
| 0.8 | 0.057 | 2.502 | 44 | 0.072 | 2.646 | 22 | 0.097 | 2.773 | 15 | 0.098 | 2.799 | 12 |
| 0.9 | 0.050 | 2.495 | 110 | 0.050 | 2.545 | 55 | 0.050 | 2.574 | 40 | 0.070 | 2.699 | 31 |
| 1.1 | 0.050 | 2.495 | 30 | 0.050 | 2.545 | 30 | 0.050 | 2.574 | 26 | 0.050 | 2.587 | 23 |
| 1.2 | 0.050 | 2.495 | 11 | 0.050 | 2.545 | 12 | 0.050 | 2.574 | 10 | 0.050 | 2.587 | 9 |
| 1.3 | 0.050 | 2.495 | 7 | 0.050 | 2.545 | 7 | 0.050 | 2.574 | 6 | 0.050 | 2.587 | 6 |
| 1.4 | 0.050 | 2.495 | 5 | 0.050 | 2.545 | 5 | 0.250 | 2.948 | 5 | 0.050 | 2.587 | 4 |
| 1.5 | 0.050 | 2.495 | 4 | 0.050 | 2.545 | 4 | 0.273 | 2.955 | 3 | 0.102 | 2.810 | 3 |
| 1.6 | 0.050 | 2.495 | 3 | 0.257 | 2.883 | 3 | 0.050 | 2.574 | 3 | 0.293 | 2.998 | 2 |
| 1.7 | 0.050 | 2.495 | 3 | 0.050 | 2.545 | 3 | 0.200 | 2.922 | 3 | 0.134 | 2.877 | 2 |
| 1.8 | 0.050 | 2.495 | 3 | 0.320 | 2.893 | 2 | 0.127 | 2.839 | 2 | 0.764 | 2.958 | 1 |
| 1.9 | 0.050 | 2.495 | 2 | 0.176 | 2.843 | 2 | 0.050 | 2.574 | 2 | 0.500 | 3.003 | 1 |
| 2.0 | 0.050 | 2.495 | 2 | 0.050 | 2.545 | 2 | 0.569 | 2.941 | 1 | 0.366 | 3.009 | 1 |

$MRL_0$ = 200. Tables 1 and 2 help practitioners to make a quick selection of the optimal parameters. For example, if a practitioner desires to construct a $S^2$-EWMA chart that is optimal for a process variance shift of $\tau$ = 0.5 ($\sigma_0$ has decreased by 50%, i.e. a process improvement), when $n$ = 3 and $MRL_0$ = 370, the associated optimal combination of parameters is ($\lambda^*$ = 0.090, $K^*$ = 2.807) and the minimum $MRL_1$ for this shift is 13. Similarly, for $\tau$ = 1.5 ($\sigma_0$ has increased by 50%), when $n$ = 5 and $MRL_0$ = 200, the corresponding optimal combination of parameters is ($\lambda^*$ = 0.050, $K^*$ = 2.545) and the minimum $MRL_1$ is 4. As illustrated in Tables 1 and 2, generally, smaller values of $\lambda$ are more likely to be optimal in detecting shifts (even for large shifts) in the process variance. Tables 1 and 2 also indicate that the smoothing constant $\lambda$ = 0.050 seems to be a good choice to obtain the minimum $MRL_1$ in most of

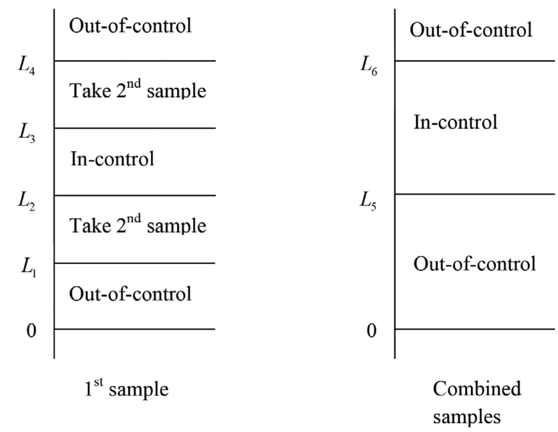


FIGURE 1. A graphical view of the DS $S^2$ chart

TABLE 3. MRL comparison between the DS $S^2$, $S$ and $S^2$-EWMA charts, for $ASS_0$ or $n = 3, 5, 7, 9$ and $MRL_0 = 200$

| | DS $S^2$ Chart | | | | $S$ Chart | | | | $S^2$-EWMA Chart | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $L_1 = 0.00001$<br>$L_2 = 0.085$<br>$L_3 = 5.120$<br>$L_4 = 11.30$<br>$L_5 = 0.024$<br>$L_6 = 10.60$<br>$n_1 = 2$<br>$n_2 = 4$ | $L_1 = 0.010$<br>$L_2 = 0.287$<br>$L_3 = 4.200$<br>$L_4 = 4.900$<br>$L_5 = 0.097$<br>$L_6 = 4.750$<br>$n_1 = 4$<br>$n_2 = 6$ | $L_1 = 0.010$<br>$L_2 = 0.339$<br>$L_3 = 2.742$<br>$L_4 = 3.700$<br>$L_5 = 0.196$<br>$L_6 = 3.570$<br>$n_1 = 6$<br>$n_2 = 8$ | $L_1 = 0.010$<br>$L_2 = 0.399$<br>$L_3 = 2.852$<br>$L_4 = 3.200$<br>$L_5 = 0.269$<br>$L_6 = 2.910$<br>$n_1 = 8$<br>$n_2 = 10$ | | | | | | | | |
| $\tau$ | $ASS_0 = 3$ | $ASS_0 = 5$ | $ASS_0 = 7$ | $ASS_0 = 9$ | $n = 3$ | $n = 5$ | $n = 7$ | $n = 9$ | $n = 3$ | $n = 5$ | $n = 7$ | $n = 9$ |
| 0.5 | 48 | 15 | 4 | 2 | 101 | 28 | 11 | 5 | 12 | 6 | 4 | 3 |
| 0.6 | 79 | 35 | 10 | 5 | 144 | 56 | 25 | 13 | 15 | 8 | 5 | 4 |
| 0.7 | 116 | 76 | 28 | 13 | 196 | 100 | 57 | 35 | 24 | 12 | 8 | 6 |
| 0.8 | 152 | 150 | 79 | 43 | 252 | 167 | 116 | 84 | 44 | 22 | 15 | 12 |
| 0.9 | 185 | 232 | 192 | 132 | 274 | 240 | 208 | 181 | 110 | 55 | 40 | 31 |
| 1.1 | 164 | 100 | 78 | 72 | 105 | 90 | 80 | 72 | 30 | 30 | 26 | 23 |
| 1.2 | 113 | 43 | 31 | 25 | 52 | 38 | 30 | 24 | 11 | 12 | 10 | 9 |
| 1.3 | 71 | 22 | 14 | 11 | 29 | 19 | 14 | 11 | 7 | 7 | 6 | 6 |
| 1.4 | 46 | 13 | 8 | 6 | 18 | 10 | 7 | 6 | 5 | 5 | 5 | 4 |
| 1.5 | 32 | 9 | 5 | 4 | 12 | 7 | 5 | 4 | 4 | 4 | 3 | 3 |
| 1.6 | 23 | 6 | 4 | 3 | 8 | 5 | 3 | 3 | 3 | 3 | 3 | 2 |
| 1.7 | 18 | 5 | 3 | 2 | 6 | 4 | 3 | 2 | 3 | 3 | 3 | 2 |
| 1.8 | 14 | 4 | 2 | 2 | 5 | 3 | 2 | 2 | 3 | 2 | 2 | 1 |
| 1.9 | 11 | 3 | 2 | 1 | 4 | 2 | 2 | 1 | 2 | 2 | 2 | 1 |
| 2.0 | 9 | 2 | 2 | 1 | 4 | 2 | 1 | 1 | 2 | 2 | 1 | 1 |

the cases. The accuracies of all the entries in Tables 1 and 2 have been verified with simulation using SAS.

The DS $S^2$ chart allows a quick detection of small process shifts while the traditional $S$ chart is capable of detecting large process shifts quickly (He & Grigoryan 2003; Khoo 2004). The $S^2$-EWMA chart is compared with both the DS $S^2$ chart and the $S$ chart, where $MRL_0 = 200$ and 370, $\tau \in (0.5, 2)$ and $n = \{3, 5, 7, 9\}$ are considered. Note that instead of a fixed sample size, $n$, the DS $S^2$ chart uses the average sample size (ASS) due to its adaptive feature. The design of the DS $S^2$ chart depends on eight parameters, i.e. the sizes of the first and second samples ($n_1$ and $n_2$), limits associated with the first sample ($L_1$, $L_2$, $L_3$ and $L_4$) and limits associated with the second sample ($L_5$ and $L_6$), as shown in Figure 1. There are three possibilities after the first sample is taken, i.e. the process is in-control if the variance of the first sample $S_1^2 \in (L_2, L_3)$; the process is out-of-control if $S_1^2 \in [(0, L_1) \cup (L_4, \infty)]$; and a second sample is taken if $S_1^2 \in [(L_1, L_2) \cup (L_3, L_4)]$. The $MRL_1$s are computed for different values of $\tau$, $n$ and $ASS_0$ for all the three charts in Table 3. For an equal footing comparison, the classical Shewhart $S$ chart, DS $S^2$ chart and the $S^2$-EWMA chart are designed for the magnitude of shifts $\tau \in (0.5, 2)$. As the results for $MRL_0 = 370$ show similar trend to that for $MRL_0 = 200$, only the results for $MRL_0 = 200$ are presented in Table 3.

The $S^2$-EWMA chart is superior to the DS $S^2$ and $S$ charts. This is always the case regardless of the value of $\tau$. It is clearly seen that almost all the $MRL_1$s in Table 3 for the $S^2$-EWMA chart are less than or equal to the corresponding ones of the DS $S^2$ and $S$ charts. The difference is particularly remarkable for process improvement ($\tau < 1$). For example, for $n = 3$, when one wishes to detect a 50, 40 or 30% decrease in the variance (i.e. $\tau = 0.5$, 0.6 or 0.7), the $MRL_1$s for the $S^2$-EWMA chart are 12, 15 or 24, while the corresponding $MRL_1$s for the $S$ chart are 101, 144 or 196, while that for the DS $S^2$ chart are 48, 79 and 116, respectively. From Table 3, it is obvious that there is almost no difference between the $MRL_1$s for the $S^2$-EWMA, DS $S^2$ and $S$ charts when both the sample size and an increase in the variance are large. It is evident that when $n = 7$ or 9, with at least a 70% increase in the variance (i.e. $\tau = 1.7$, 1.8, 1.9 or 2.0), the $MRL_1$s for the $S^2$-EWMA, DS $S^2$ and $S$ charts are nearly the same. From the above discussion, it is clear that the $S^2$-EWMA chart is superior to the DS $S^2$ and S charts.

## CONCLUSION

This paper presents an optimal design of the $S^2$-EWMA chart to monitor the process variance, based on the MRL criterion, instead of relying solely on the ARL criterion. As explained in the Introduction section, the MRL gives more information compared to the ARL. The MRL is also more readily understood by practitioners when it comes to a highly skewed run length distribution. This paper complements the work of Castagliola (2005), where the design of the $S^2$-EWMA chart is based on the ARL. Thus, it is timely to provide the optimal design of the $S^2$-EWMA chart based on MRL to practitioners. A comparison of the MRL performance of the $S^2$-EWMA (derived via the Markov chain approach), DS $S^2$ and the $S$ charts show that the $S^2$-EWMA chart outperforms the DS $S^2$ and $S$ charts, for detecting changes in the process variance. Lastly, the CUSUM version of the $S^2$ charting method using the MRL criterion is a topic worthy of further research.

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## APPENDIX (MARKOV CHAIN TECHNIQUE)

The MRL of the $S^2$-EWMA chart can be evaluated using the Markov chain approximation. This discrete-time Markov chain approach, originally proposed by Brook and Evans (1972), is flexible and relatively easy to use. This procedure divides the interval between the upper control limit (UCL) and lower control limit (LCL) into $p = 2m + 1$ sub-intervals, each of width $2\delta$ (Figure A1), where $\delta = \frac{UCL - LCL}{2p}$. The control charting statistic in (3) is said to be in transient state $j$ at time $k$ if $H_j - \delta < Z_k < H_j + \delta$, for $j = -m, \ldots, -1, 0, +1, \ldots, +m$, where $H_j$ represents the midpoint of the $j$th subinterval. The control charting statistic is in the absorbing state if $Z_k$ falls outside the control limits. The process is assumed to be in-control whenever $Z_k$ is in a transient state and is assumed to be out-of-control whenever $Z_k$ is in the absorbing state.

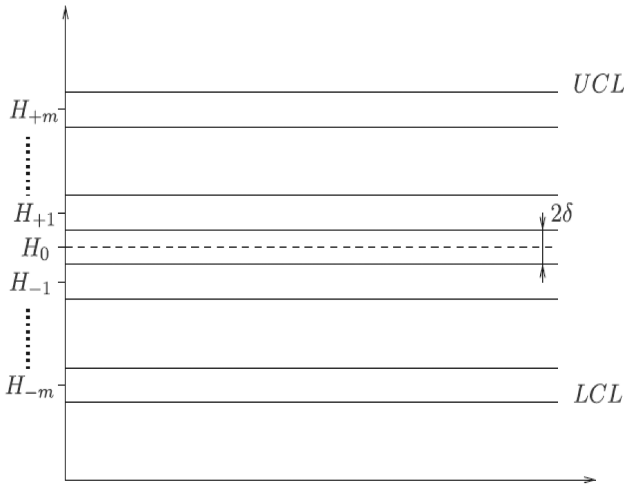


FIGURE A1. Interval between LCL and UCL divided into $p = 2m + 1$ sub-intervals of width $2\delta$

Let $M$ be the run length of a control scheme, i.e. $M$ represents the number of steps required until the process reaches the absorbing state. Here, $M$ is a discrete phase type random variable, i.e. its distribution $f(m)$, for $m = 1, 2, \ldots$, corresponds to the distribution of the first passage time to the absorbing state of a Markov chain with finitely many states, where all states are transient, except one which is absorbing. Then the cumulative distribution function (cdf) of the run length, $M$ of this control scheme is (Brook & Evans 1972)

$$\Pr(M \le m) = \mathbf{s}^T(\mathbf{I} - \mathbf{Q}^m)\mathbf{1}, \tag{A1}$$

where matrix $\mathbf{Q}$ is the transition probability matrix for the transient states (after removing the absorbing state), $\mathbf{I}$ is the $(p \times p)$ identity matrix, $\mathbf{1}$ is a vector with each of its $p$ elements equal to unity and $\mathbf{s}$ is the initial probability column vector having $(2m+1)$ elements, with a single element corresponding to the initial state equals one and zero elsewhere. The transition probability matrix $\mathbf{Q}$ contains the one-step transition probabilities. The generic element $p_{i,j}$ of $\mathbf{Q}$ represents the probability that the control statistic goes from state $i$ to state $j$ in one step. As stated by Lucas and Saccucci (1990), in order to approximate this probability, it is assumed that the control statistic is equal to $H_j$ whenever it is in state $j$, i.e.

$$p_{i,j} \cong \Pr\left[\frac{(H_j - \delta) - (1-\lambda)H_i}{\lambda} < T_k < \frac{(H_j + \delta) - (1-\lambda)H_i}{\lambda}\right]. \tag{A2}$$

Introducing the cdf of the random variable $T_k$, (A2) can be rewritten as

$$p_{i,j} \cong F_{T_k}\left[\frac{(H_j + \delta) - (1-\lambda)H_i}{\lambda}\right] - F_{T_k}\left[\frac{(H_j - \delta) - (1-\lambda)H_i}{\lambda}\right]. \tag{A3}$$

The cdf $F_{T_k}\left(t|n,\tau\right)$ of $T_k$ is defined for $t \geq A(n) + B(n)ln[C(n)]$ and it is equal to

$$F_{T_k}\left(t|n,\tau\right) = F_G\left\{\exp\left(\frac{t-A(n)}{B(n)}\right) - C(n)\left|\frac{n-1}{2},\frac{2\tau^2}{n-1}\right.\right\}, \tag{A4}$$

where $F_G(x|u, v)$ is the cdf of the gamma $G(u, v)$ distribution.

Thus, in our case, the generic element $Q_{i,j}$ of matrix $\mathbf{Q}$ of transient probabilities is equal to

$$\begin{aligned} Q_{ij} = F_G&\left\{\exp\left(\frac{1}{B(n)}\left[\frac{(H_j+\delta)-(1-\lambda)H_i}{\lambda} - A(n)\right]\right) - C(n)\left|\frac{n-1}{2},\frac{2\tau^2}{n-1}\right.\right\} - \\ &F_G\left\{\exp\left(\frac{1}{B(n)}\left[\frac{(H_j-\delta)-(1-\lambda)H_i}{\lambda} - A(n)\right]\right) - C(n)\left|\frac{n-1}{2},\frac{2\tau^2}{n-1}\right.\right\}. \end{aligned} \tag{A5}$$

The generic element $s_j$ of vector $\mathbf{s}$ of initial probabilities is equal to

$$s_j = \begin{cases} 1 & \text{if} \quad H_j - \delta < Z_0 < H_j + \delta \\ 0 & \text{otherwise} \end{cases}, \tag{A6}$$

for $j = -m, -m+1, \ldots, 0, \ldots, +m$, with $Z_0$ evaluated using (10). Consequently, this vector contains only a single element equal to 1, with the remaining $2m$ entries equal to 0.

Then the $100\gamma$ ($0 < \gamma < 1$) percentage points of the run length distribution corresponding to desired values of $n$ and $\delta$ can be determined as the value $m_\gamma$ such that (Gan 1993a)

$$\Pr(M \leq m_\gamma - 1) \leq \gamma \quad \text{and} \tag{A7a}$$

$$\Pr(M \leq m_\gamma) > \gamma. \tag{A7b}$$

If $\gamma = 0.5$, the MRL can be computed. Equations (A7a) and (A7b) enable the computation of any percentage points of the run length distribution.