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A Univariate Rational Quadratic Trigonometric Interpolating Spline to Visualize Shaped Data

(Suatu Nisbah Kuadratik Trigonometri Univariat Menginterpolasikan Splin untuk Menggambarkan Data Berbentuk)

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ABSTRACT

This study was concerned with shape preserving interpolation of 2D data. A piecewise C^1 univariate rational quadratic trigonometric spline including three positive parameters was devised to produce a shaped interpolant for given shaped data. Positive and monotone curve interpolation schemes were presented to sustain the respective shape features of data. Each scheme was tested for plentiful shaped data sets to substantiate the assertion made in their construction. Moreover, these schemes were compared with conventional shape preserving rational quadratic splines to demonstrate the usefulness of their construction.

Keywords: Continuity; interpolation; rational trigonometric spline; shape preserving

ABSTRAK

Kajian ini adalah berkaitan dengan pemeliharaan bentuk interpolasi data 2D. Suatu cebisan splin nisbah kuadratik trigonometri univariat termasuk tiga parameter positif dirangka untuk menghasilkan interpolasi yang terbentuk daripada bentuk data yang diberikan. Skim lengkung interpolasi positif dan berekanada dibentangkan bagi mengekalkan ciri bentuk data masing-masing. Setiap skim diuji daripada banyak bentuk set data untuk menyokong kenyataan yang dibuat pembinaan berdasarkan skim tersebut. Selain itu, skim ini dibandingkan dengan bentuk konvensional splin nisbah kuadratik untuk menunjukkan kegunaan pembinaannya.

Kata kunci: Interpolasi; keselanjaran; pemeliharaan bentuk; splin nisbah trigonometri

INTRODUCTION

One of the key problems encountered in Computer Aided Geometric Design (CAGD), in particular in scientific computation, is the interpolation of data. Fitting a smooth curve or surface through a set of data points has many applications, for instance, design of airplanes, ships, space vehicles and machine parts to name just a few. The need for shape preserving interpolation techniques arises when the data carries some inherent shape features. In such cases only that interpolant is preferred which preserves the shape features of data. On the whole, a shaped data arising from some scientific phenomenon can be categorized into three fundamental types namely, positive, monotone and convex. Since polynomial splines do not meet this challenge, so some restriction may be enforced to obtain the desired shape preserving splines (Kvasov 2000).

Introduction of rational splines with shape parameters helped a great deal in coping with the difficulties met in case of polynomial splines. These shape parameters have been chosen so that the rational interpolant reflects the shape characteristics of data. Thus these parameters serve as a designer's intuitive tools for manipulating the shape of the curve. Plenty of works have been reported in literature in this area of research (Fritsch & Carlson 1980; Goodman 2001; Hussain et al. 2010; Hussain & Ali 2006; Sarfraz 2002; Sarfraz & Hussain 2006; Sarfraz et al. 2012; Schmidt & Hess 1987; Schumaker 1983) and the references therein, presenting several types of rational interpolating splines with shape parameters.

A recent development in CAGD is the introduction of shape preserving trigonometric splines based on trigonometric polynomials and a blending of algebraic and trigonometric polynomials. By and large, trigonometric splines are believed to be unfit to capture shape characteristics of data mainly due to their oscillatory behaviour. This drawback was removed by incorporating shape control factors in their blending functions which were used to govern any such act of these splines. Since this is comparatively a new study in the field of shape preserving data visualization, therefore only few researchers have worked (Bashir & Ali 2013; Ibraheem et al. 2012; Pan & Wang 2007; Zhu et al. 2012) on introducing cubic rational trigonometric splines with shape parameters to visualize shaped data.

In this paper, a novel rational quadratic trigonometric spline, based on Bézier-like blending functions (Bashir et al. 2013), is introduced to develop two schemes to preserve the shape of positive and monotone data. Generally, Bézier functions do not carry the underlying geometric feature of data. To overcome this deficiency, three positive parameters were integrated in the construction of the aforesaid spline. These parameters help in generating smooth curves carrying the inherent shape properties of the data and thus staving off extra knots at the points where the shape of data was lost. Since this spline is a three point based technique, it is more robust and requires less computations. The developed schemes are local, C^1 and are applicable to both uniform and non uniform data sets. These schemes were examined for a variety of shaped data sets and it has been turned up that they are effective in practice. Moreover, these schemes were compared with ordinary shape preserving rational quadratic splines as to the best of author's knowledge, there does not exist any such shape preserving univariate rational quadratic trigonometric spline in literature.

PIECEWISE C^1 RATIONAL QUADRATIC TRIGONOMETRIC SPLINE

This section discusses the formation of a piecewise C^1 rational quadratic trigonometric spline using quadratic trigonometric blending functions defined in Bashir et al. (2013).

Suppose that $\left\{\left(x_i, y_i\right)_{i=0}^{n}\right\}$ is a given set of interpolating points with $\sigma_0 = x_0 < x_1 < x_2 <,...,x_n = \sigma_n$. Let m_i be the slopes of tangent vectors at the partition points (knots) x_i. For $u = \frac{\pi}{2}\frac{x - x_i}{h_i}$ with $h_i = x_{i+1} - x_i$, a piecewise rational quadratic trigonometric spline $P(x) \in C^1[\sigma_0, \sigma_n]$ on each subinterval $I = [x_i, x_{i+1}]$, $i = 0(1)n - 1$ with three positive parameters λ_i^0, λ_i^1 and λ_i^2 is defined as:

$$P(x) \equiv P_i(x) = \sum_{j=0}^{2} M_j g_j, \tag{1}$$

where $g_j = \frac{b_j}{\sum_{j=0}^{2}\lambda_i^j b_j}$ are the rational basis functions with b_J [15]:

$$\begin{aligned} b_0(u) &= (1 - \sin u)(1 - \alpha \sin u) \\ b_1(u) &= 1 - b_0(u) - b_2(u) \\ b_2(u) &= (1 - \cos u)(1 - \beta \cos u), \end{aligned} \tag{2}$$

and M_j are unknowns to be determined.

The necessary conditions for the spline (1) to be of class $C^1[\sigma_0, \sigma_n]$ are:

$$P(x_k) = y_k \text{ and } P'(x_k) = m_k, \quad k = i, i + 1. \tag{3}$$

Thus (3) indicates that $P(x)$ interpolates both the data points and the tangent vectors at knots. If not given, these slopes were determined from the data using some approximation methods.

Using the continuity conditions, the values of M_j, $j =$ 0, 1. 2 are:

$$M_0 = \lambda_i^0 y_i,\ M_1 = \lambda_i^1 y_i + \frac{2m_i h_i \lambda_i^0}{(1+\alpha)\pi},\ M_2 = \lambda_i^2 y_{i+1}, \tag{4}$$

with these values of M_j, (1) takes the form

$$P(x) \equiv P_i(x) = \frac{p_i(u)}{q_i(u)}, \tag{5}$$

where,

$$p_i(u) = \lambda_i^0 y_i b_0 + \left(\lambda_i^1 y_i + \frac{2m_i h_i \lambda_i^0}{(1+\alpha)\pi}\right) b_1 + \lambda_i^2 y_{i+1} b_2.$$

$$q_i(u) = \lambda_i^0 b_0 + \lambda_i^1 b_1 + \lambda_i^2 b_2$$

Also, $P(x)$ will preserve Hermite C^1 continuity at the knots if and only if $P'(x_{i+1}) = m_{i+1}$. Which yields,

$$\lambda_i^1 = \frac{2\left(\left(1+\beta\right)\lambda_i^0 m_i + \left(1+\alpha\right)\lambda_i^2 m_{i+1}\right)}{\left(1+\alpha\right)\left(1+\beta\right)\pi\Delta_i},\ \ \alpha, \beta \in (-1, 1] \tag{6}$$

provided that $\Delta_i = \frac{y_{i+1} - y_i}{h_i} \neq 0$. If $\Delta_i = 0$, then $P_i(x) = y_i$, $\forall\ x \in [x_i, x_{i+1}]$.

POSITIVE CURVE INTERPOLATION

In this section, the proposed piecewise C^1 rational quadratic trigonometric spline is analysed to preserve the positivity of positive data. It is to mention that the spline under discussion does not produce a positivity preserving curve by merely using the constrained value of λ_i^1 defined by (6). Thus some extra effort is needed to hold the geometric attribute of the data.

Theorem 1 Let $\left\{\left(x_i, y_i\right)_{i=0}^{n}\right\}$ be a given positive data set. On each subinterval I_i, the piecewise C^1 rational quadratic trigonometric spline (5) is positive if,

$$\lambda_i^1 > \max\left\{0, -\frac{2m_i h_i \lambda_i^0}{(1+\alpha)\pi y_i}\right\}.$$

Proof Since the data set is given to be positive, for $xi < x_{i+1}$, $i = 0(1)n - 1$, it follows that

$$y_i > 0,\ \ i = 0(1)n.$$

Thus the spline (5) is positive if $p_i(u)$ and $q_i(u)$ observe the same signs. The positivity of blending functions b_j and parameters $\lambda_i^j, j = 0, 1, 2$ guarantee that $q_i(u) > 0$, thus the problem of generating a positivity preserving interpolating curve reduces to impose restrictions on the coefficients of $p_i(u)$ that make is positive.

$$p_i(u) > 0 \text{ if, } \quad \lambda_i^1 > -\frac{2m_i h_i \lambda_i^0}{(1+\alpha)\pi y_i}.$$

Since $\lambda_i^1 > 0$, therefore for positivity preserving spline

$$\lambda_i^1 > \max\left\{0, \frac{2m_i h_i \lambda_i^0}{(1+\alpha)\pi y_i}\right\}. \tag{7}$$

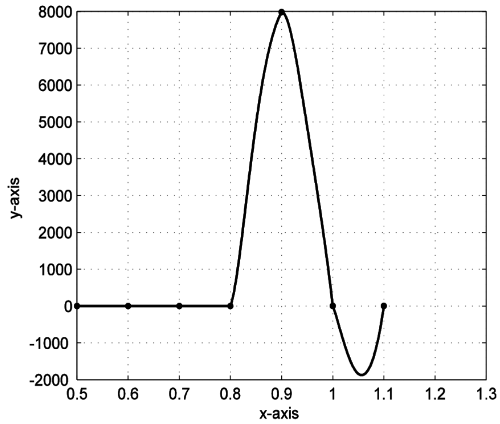
GRAPHICAL DEMONSTRATION

Example 1 With $\alpha = -0.8$ and $\beta = -0.2$, Figure 1 shows curves drawn by using a positive data set given in Table 1, taken from Gerald and Wheatley (2003). Figure 1(a) displays the curve generated by selecting the values of parameters randomly. The non-positive trend of the curve is clearly visible. On the other hand, the curve in Figure 1(b) is constructed by positivity preserving piecewise C^1 rational quadratic trigonometric spline assuming $\lambda_i^0 = \lambda_i^2 = 1.2$. This figure shows that the resulting curve is smooth and preserves the geometric feature of the data. Figure 2 presents a comparison of the curve generated by proposed scheme with Hussain et al. (2007) (RQ-function; in short). The comparison shows that the proposed scheme produces graphically smooth shape preserving curve and can be used as an alternative to existing scheme.

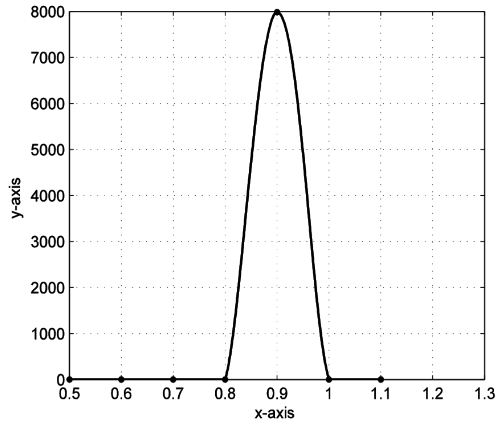
Example 2 This example presents the curves drawn by employing the positive data set given in Table 2. The curves in Figure 3 are generated by taking $\alpha = 1$ and $\beta = -0.5$. At first, piecewise rational quadratic trigonometric spline

TABLE 1. Data for Figure 1

i	1	2	3	4	5	6	7
x_i	0.1	0.2	0.3	0.4	0.5	0.6	0.7
y_i	0.4804	0.5669	0.7262	0.1000	7985	0.8658	0.9281



(a) Non-positive curve



(b) Positivity preserving curve

FIGURE 1. Curves with (a) C^1 RQT-spline and (b) C^1 positivity preserving RQT-spline

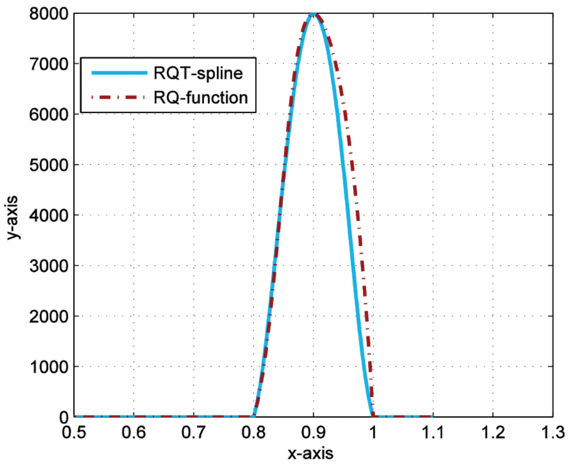
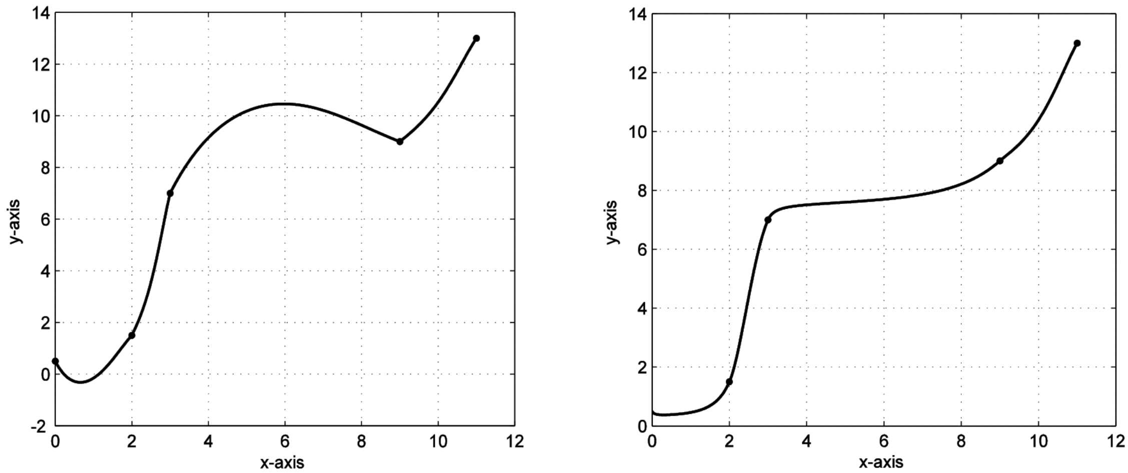


FIGURE 2. Comparison with RQ-function

TABLE 2. Data for Figure 3

i	1	2	3	4	5
x_i	0	2.0	3.0	7.0	11.0
y_i	0.5	1.5	7.0	9.0	13.0



(a) Non-positive curve (b) Positivity preserving curve

FIGURE 3. Curves with (a) C^1 RQT-spline (b) C^1 positivity preserving RQT-spline

is employed to construct the curve presented in Figure 3(a). The curve showed the non-positive behaviour of the positive data. The same data is interpolated by implementing positivity preserving scheme in Figure 3(b) with the values of parameters set as $\lambda_i^0 = 0.6$ and $\lambda_i^2 = 1.0$. A comparison of the curve constructed by proposed scheme with existing scheme (Hussain et al. 2007) is given in Figure 4. The difference in smoothness is clearly manifested.

MONOTONE CURVE INTERPOLATION

This section is dedicated to the construction of monotone curve interpolation scheme when the data under consideration

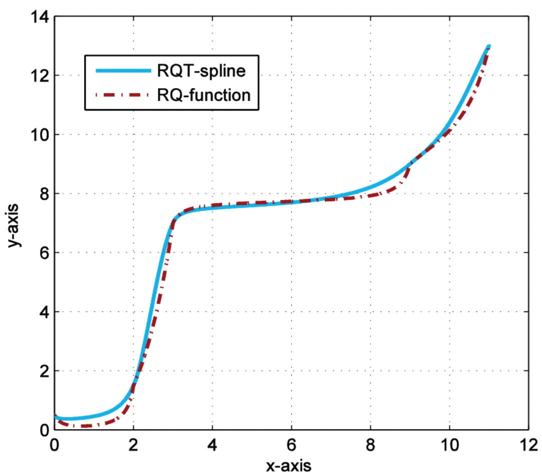


FIGURE 4. Comparison with RQ-function

is monotone. Sufficient conditions are derived for two parameters appearing in piecewise C^1 rational quadratic trigonometric spline while keeping one free.

Theorem 2 For a given monotone data set $\left\{(x_i, y_i)_{i=0}^{n}\right\}$, spline defined in (5) generates a monotone curve over an arbitrary interval $[x_i, x_{i+1}]$ if,

$$\lambda_i^1 \geq \max\left\{0, \frac{2m_i\lambda_i^0}{(1+\alpha)\pi\Delta_i}\right\} \text{ and } \lambda_i^2 > \max\left\{0, \frac{2m_i\lambda_i^0}{(1+\alpha)\pi\Delta_i}\right\}.$$

Proof Without loss of generality, assume that the data set is monotonically increasing.

That is,

$$y_{i+1} > y_i \text{ or } \Delta_i = \frac{y_{i+1} - y_i}{h_i} \geq 0, \quad i = 0(1)n-1.$$

To preserve the monotonicity, the necessary conditions on the slopes are:

$$m_i = 0, \quad \text{if } \Delta_i = 0$$
$$m_i > 0, \quad \text{otherwise.}$$

In case $\Delta_i = 0$, the spline (5) reduces to $P_i(x) = y_i$, $\forall x \in [x_i, x_{i+1}]$ and thus the interpolating curve is trivially monotone. For the case $\Delta_i \neq 0$, $P_i(x)$ increases monotonically if and only if,

$$P_i'(x) > 0, \quad \forall x \in [x_i, x_{i+1}].$$

For $x \in [x_i, x_{i+1}]$, $P_i'(x)$, is given by:

$$P_i'(x) = \frac{\pi}{2(q_i(u))^2}\left\{\begin{array}{l}(\alpha\cos u(1-\sin u)+\cos u(1-\alpha\sin u))\\(A_1+A_2+A_3)\\+(\beta\sin u(1-\cos u)+\sin u(1-\beta\cos u))\\(A_4+A_5)\end{array}\right\}, \quad (8)$$

with, $A_1 = N_1(1-b_2)$, $A_2 = N_2 b_2$, $A_3 = N_3 b_2$, $A_4 = N_4(1-b_0)$, $A_5 = N_5 b_0$.

such that,

$$N_1 = \frac{2m_i(\lambda_i^0)^2}{(1+\alpha)\pi},\ N_2 = \lambda_i^0\lambda_i^2\Delta_i,\ N_3 = \left(-\lambda_i^1\Delta_i + \frac{2m_i\lambda_i^0}{(1+\alpha)\pi}\right)\lambda_i^2,$$

$$N_4 = \left(\lambda_i^1\Delta_i - \frac{2m_i\lambda_i^0}{(1+\alpha)\pi}\right)\lambda_i^2,\ N_5 = \left(\lambda_i^2\Delta_i - \frac{2m_i\lambda_i^0}{(1+\alpha)\pi}\right)\lambda_i^0.$$

The positivity of blending functions b_j and the parameters $\lambda_i^j, j = 0, 1, 2$ declare that the positivity of $P_i'(x)$ is subject to that of N_k, $k = 0, 1, \ldots, 5$ only.

$N_1, N_2 > 0$ evidently. However, $N_3, N_4, N_5 > 0$ if,

$$\lambda_i^1 > \frac{2m_i\lambda_i^0}{(1+\alpha)\pi\Delta_i},\ \lambda_i^1 < \frac{2m_i\lambda_i^0}{(1+\alpha)\pi\Delta_i} \text{ and } \lambda_i^2 > \frac{2m_i\lambda_i^0}{(1+\alpha)\pi\Delta_i},$$

using $\lambda_i^1 > 0, j = 0, 1, 2$, piecewise C^1 rational trigonometric spline preserves the monotonicity of monotone data if,

$$\lambda_i^1 \geq \max\left\{0, \frac{2m_i\lambda_i^0}{(1+\alpha)\pi\Delta_i}\right\} \text{ and } \lambda_i^2 > \max\left\{0, \frac{2m_i\lambda_i^0}{(1+\alpha)\pi\Delta_i}\right\}.$$

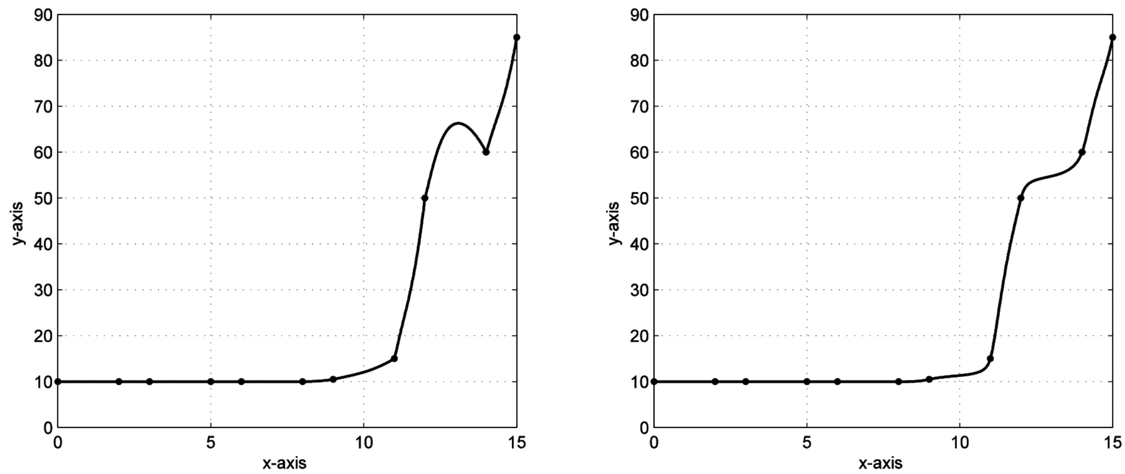
GRAPHICAL DEMONSTRATION

Example 3 Consider a monotone data set given in Table 3 borrowed from Akima (1970) while taking $\alpha = -0.7$ and $\beta = 0.9$. The resulting curves are displayed in Figure 5. Curve in Figure 5(a) violates monotonicity of the data. In order to hold the built in feature of the data, the interpolating curve is generated by using the constrained values of λ_i^1 and λ_i^2, with $\lambda_i^0 = 1.3$. This treatment produces a monotone and smooth curve which is presented in Figure 5(b). Figure 6 presents a comparison of the rational quadratic spline with the aforesaid scheme.

Example 4 In another example, the curves are constructed for a monotone data set given in Table 4. The curves interpolating this data was created with the help of piecewise rational quadratic interpolating spline by assuming $\alpha = 1$ and $\beta = 0$ are displayed in Figure 7. Figure 7(a) shows a non-monotonicity preserving curve generated by taking arbitrary values of parameters. On the contrary Figure 7(b) presents that the curve holds monotonicity when the developed monotone curve interpolation scheme is used for curve rendering with $\lambda_i^0 = 1.0$. This proves the significance of the scheme. A comparison of the proposed

TABLE 3. Data for Figure 5

i	1	2	3	4	5	6	7	8	9	10	11
x_i	0.0	2.0	3.0	5.0	6.0	8.0	9.0	11.0	12.0	14.0	15.0
y_i	10.0	10.0	10.0	10.0	10.0	10.0	10.5	15.0	50.0	60.0	85.0



(a) Non- monotone curve

(b) Monotone curve

FIGURE 5. Curves with (a) C^1 RQT-spline and (b) C^1 monotonicity preserving RQT-spline

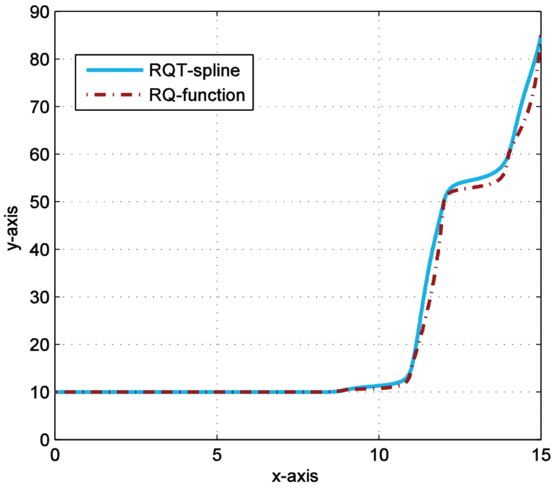
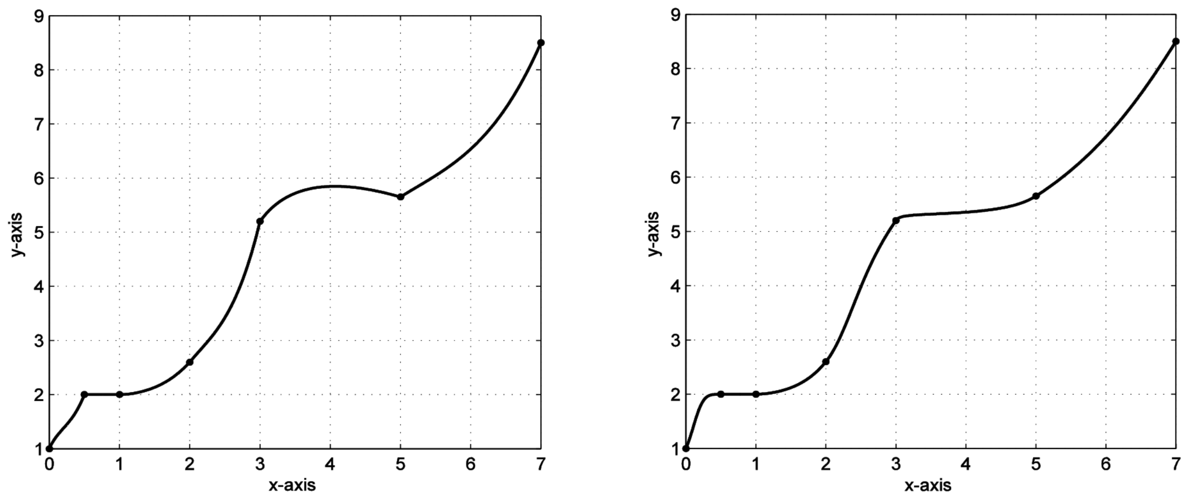


FIGURE 6. Comparison with RQ-function

TABLE 4. Data for Figure 6

i	1	2	3	4	5	6	7
x_i	0.0	0.5	1.0	2.0	3.0	5.0	7.0
y_i	1.0	2.0	2.0	2.6	5.2	5.65	8.5



(a) Non- monotone curve

(b) Monotone curve

FIGURE 7. Curves with (a) C^1 RQT-spline (b) C^1 monotonicity preserving RQT-spline

scheme with the aforementioned RQ-function is displayed in Figure 8.

CONCLUSION

Shape preserving scientific data visualization curve construction technique is discussed in this paper. Rational quadratic trigonometric blending function with two shape parameters are used to generate a family of piecewise C^1 univariate rational quadratic trigonometric spline to develop positive and monotone, constrained and monotone curve interpolation schemes. Three positive parameters are included in the description of this spline. These parameters play a dual role. Firstly, they help to preserve the geometrical attributes inherited in the data and secondly they enhance the smoothness of the resulting curve.

The presented schemes carry some characteristics: Each technique is based on trigonometric functions, therefore provides an alternative to conventional shape preserving interpolating splines. In addition, quadratic functions are simple and carry less computation efforts. The schemes are local and preserve first order Hermite

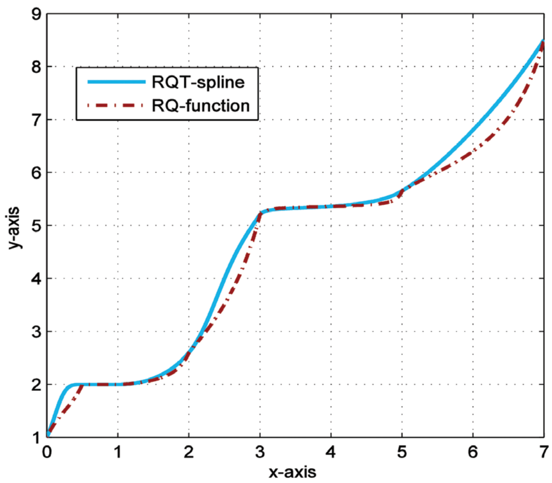


FIGURE 8. Comparison with RQ-function

continuity which ensures the smoothness at the knots. Each scheme is applicable to both equally and unequally spaced data. Neither they depend on the function values nor on the slopes of tangent vectors, rather these are computed from the data (if not given) by using some numerical methods. The constructed schemes are judged for a variety of data sets and compared with the existing rational quadratic splines. It has been demonstrated that the proposed schemes produce smoother curves than the latter.

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