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A Bayesian Approach to the One Way ANOVA under Unequal Variance

(Pendekatan Bayesian kepada ANOVA Sehala di bawah Varians tak Sama)

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ABSTRACT

This study involves testing the equality of several normal means under unequal variances, which is the setup of one-way analysis of variances (one-way ANOVA). Several tests are available in the literature, however, most of them perform poorly in terms of type I error rate under unequal variances. In fact, Type I errors can be highly inflated for some of the commonly used tests, a serious issue that seems to have been overlooked. Even though several tests have been proposed to overcome the problem, most of them show difficulty in calculation. Accordingly, the test for ANOVA with estimation of parameters using Bayesian approach is proposed as an alternative to such tests. The proposed test is compared with four existing tests such as the original test, James's test, Welch's test and the parametric bootstrap (PB) test. Type I error rates and powers of the tests are evaluated using Monte Carlo simulation. Our results indicated that the performance of the proposed test is superior to the original test and is comparable to James's test, Welch's test and the PB test, controlling Type I error rate quite well and showing high power of the test. Our study suggested that the proposed test has high performance and should be used as an alternative to the four existing tests due to its simple formula.

Keywords: Bayesian approach; power of the test; Type I error rate; unequal variance

ABSTRAK

Kajian ini melibatkan ujian kesamaan dalam beberapa cara yang biasa di bawah varians tak sama yang merupakan persediaan varians analisis sehala (ANOVA sehala). Beberapa ujian telah sedia ada dalam penulisan ilmiah, walau bagaimanapun, tidak menunjukkan keputusan memberangsangkan daripada segi kadar ralat Jenis I di bawah varians tak sama. Malah, ralat Jenis I boleh melambung tinggi bagi sesetengah ujian yang biasa digunakan, suatu isu yang serius yang seolah-olah telah diabaikan. Walaupun beberapa ujian telah dicadangkan untuk mengatasi masalah ini, sebahagian besar menunjukkan kesukaran dalam pengiraan. Sehubungan dengan itu, ujian bagi ANOVA dengan parameter anggaran menggunakan pendekatan Bayesian dicadangkan sebagai alternatif kepada ujian tersebut. Ujian yang dicadangkan dibandingkan dengan empat ujian sedia ada seperti ujian asal, ujian James, ujian Welch dan ujian butstrap berparameter (PB). Kadar ralat Jenis I dan kuasa ujian dinilai menggunakan simulasi Monte Carlo. Keputusan kajian kami menunjukkan bahawa prestasi ujian cadangan itu lebih cemerlang berbanding ujian asal dan setanding dengan ujian James, Welch dan PB, mengawal kadar ralat Jenis I dengan baik dan menunjukkan kuasa tinggi ujian tersebut. Kajian kami menyarankan bahawa ujian cadangan mempunyai prestasi yang tinggi dan harus digunakan sebagai suatu alternatif kepada empat ujian sedia ada kerana formula yang mudah.

Kata kunci: Kadar ralat Jenis I; kuasa ujian; pendekatan Bayesian; varians tak sama

INTRODUCTION

One way analysis of variance (one-way ANOVA) is a procedure for testing the hypothesis that k population means are equal, where $k > 2$. ANOVA compares the means of the samples or groups in order to make inferences about the population means. This method is often used in scientific or medical experiments when such experiments contain several treatments. The model assumptions for ANOVA consist of independence, normality and homogeneity of variances. Sometimes homogeneity of variances can be violated. So, the F-test in ANOVA is not applicable. Box (1954) examined the effect of unequal variance on the test for one-way ANOVA and explained that the test is not serious if the groups are equal and moderately unequal variances since Type I error rate could be well controlled. In addition, Welch (1937) found that the effect of Type I error rate was small when the groups were of equal size. Assume that a random sample $X_{i1}, \ldots, X_{in_i}$ of size n_i is available for the ith population $N(\mu_i, \sigma_i^2)$, $i = 1, 2, 3, \ldots, k$. Hence, the test developed for unequal variances under the hypotheses of interest is:

$$H_0 : \mu_1 = \ldots = \mu_k \text{ vs. } H_1 : \mu_i \neq \mu_j \text{ for some } i \neq j.$$

A test statistic (Seber 1977) is given by:

$$T\left(\bar{X}_1,\ldots,\bar{X}_k;\sigma_1^2,\ldots,\sigma_k^2\right)=\sum_{i=1}^{k}\frac{n_i}{\sigma_i^2}\bar{X}_i^2-\frac{\left[\sum_{i=1}^{k}n_i\bar{X}_i/\sigma_i^2\right]^2}{\sum_{i=1}^{k}n_i/\sigma_i^2}. \qquad (1)$$

In general, the population variances σ_i^2's are unknown; in this case, a test statistic can be obtained by replacing σ_i^2 in (1) with S_i^2 and is given by:

$$T\left(\bar{X}_1,\ldots,\bar{X}_k;S_1^2,\ldots,S_k^2\right)=\sum_{i=1}^{k}\frac{n_i}{S_i^2}\bar{X}_i^2-\frac{\left[\sum_{i=1}^{k}n_i\bar{X}_i/S_i^2\right]^2}{\sum_{i=1}^{k}n_i/S_i^2}, \qquad (2)$$

where $\bar{X}_i$ and S_i^2 are maximum likelihood estimators (MLE) of population mean and population variance, respectively. The test statistic in (2) (say, the original test) has chi-square distribution with $k-1$ degrees of freedom. Therefore, the test rejects $H_0:\mu_1=\ldots=\mu_k$ at α statistical significance when,

$$T\left(\bar{X}_1,\ldots,\bar{X}_k;S_1^2,\ldots,S_k^2\right)>\chi^2_{k-1,\alpha}. \qquad (3)$$

Many researchers have proposed the test statistic to test the equality of k population means under unequal variances.

James (1951) derived a second order approximation to the distribution of the statistic $T(\bar{X}_1,\ldots\bar{X}_k;S_i^2,\ldots,S_k^2)$. The critical value, which is a function of S_i^2's based on the second-order approximation, can be expressed as follows. Let

$$Q=\sum_{i=1}^{k}\frac{1}{n_i-1}\left(1-w_i\Big/\sum_{i=1}^{k}w_i\right)^2,\quad c_s=\frac{\left(\chi^2_{k-1,\alpha}\right)^s}{(k-1)(k+1)\ldots(k+2s-3)},$$

and

$$R_{st}=\sum_{i=1}^{k}\frac{1}{(n_i-1)^s}\left(w_i\Big/\sum_{i=1}^{k}w_i\right)^t,\ \text{where } w_i=n_i/S_i^2,$$

the critical value is given as:

$$\begin{aligned}
J_\alpha={}&\chi^2_{k-1,\alpha}+\frac{1}{2}(3c_2+c_1)Q+\left\{\frac{1}{16}(3c_2+c_1)^2\left(1-\frac{k-3}{\chi^2_{k-1,\alpha}}\right)Q^2\right.\\
&+\frac{1}{2}(3c_2+c_1)\Big[\left(8R_{23}-10R_{22}+4R_{21}-6R_{12}^2+8R_{12}R_{11}-4R_{11}^2\right)\\
&+\left(2R_{23}-4R_{22}+2R_{21}-2R_{12}^2+4R_{12}R_{11}-2R_{11}^2\right)(c_1-1)\\
&+\frac{1}{4}\left(-R_{12}^2+4R_{12}R_{11}-2R_{12}R_{10}-4R_{11}^2+4R_{11}R_{10}-R_{10}^2\right)\\
&\left(3c_2-2c_1-1\right)\Big]+\left(R_{23}-3R_{22}+3R_{21}-R_{20}\right)\left(5c_3+2c_2+c_1\right)\\
&+\frac{3}{16}\left(R_{12}^2-4R_{23}+6R_{22}-4R_{21}+R_{20}\right)\left(35c_4+15c_3+9c_2+5c_1\right)\\
&+\frac{1}{16}\left(-2R_{22}+4R_{21}-R_{20}+2R_{12}R_{10}-4R_{11}R_{10}+R_{10}^2\right)\\
&\left(9c_4-3c_3-5c_2-c_1\right)\\
&+\frac{1}{4}\left(-2R_{22}+R_{11}^2\right)\left(27c_4+3c_3+c_2+c_1\right)\\
&\left.+\frac{1}{4}\left(R_{23}+R_{12}R_{11}\right)\left(45c_4+9c_3+7c_2+3c_1\right)\right\}+\left(\left(n_i-1\right)^{-3}\right).
\end{aligned} \qquad (4)$$

This test rejects H_0 when $T(\bar{x}_1,\ldots,\bar{x}_k;s_1^2,\ldots,s_k^2)>J_\alpha$, where $T(\bar{x}_1,\ldots,\bar{x}_k;s_1^2,\ldots,s_k^2)$ is the observed value of $T(\bar{X}_1,\ldots,\bar{X}_k;S_1^2,\ldots,S_k^2)$.

In 1951, Welch presented the test statistic for means under unequal variances,

$$W^*=\frac{T\left(\bar{X}_1,\ldots,\bar{X}_k;S_1^2,\ldots,S_k^2\right)\Big/(k-1)}{1+(2(k-2)/(k^2-1))\sum_{i=1}^{k}(1/(n_i-1))(1-w_i/\sum_{i=1}^{k}w_i)^2}\sim F_{f_1,f_2}\ \text{approximately} \qquad (5)$$

$$\text{where } f_1=k-1 \text{ and } f_2=\left[\frac{3}{k^2-1}\sum_{i=1}^{k}\frac{1}{n_i-1}\left(1-w_i\Big/\sum_{i=1}^{k}w_i\right)^2\right]^{-1}$$

are degree of freedom. This test rejects H_0 when $W^*>F_{f_1,f_2}$.

Algina et al. (1994) compared Welch's test and James's test under unequal variance and found that Welch's test and James's test had very similar Type I error rates and can control Type I error rates quite well.

Krishnamoorthy et al. (2007) proposed the parametric bootstrap (PB) test by improving sample mean and sample variance using bootstrap method, where the parameters are replaced by their estimates in the test statistic $T(\bar{X}_1,\ldots\bar{X}_k;S_i^2,\ldots,S_k^2)$. The PB test can be written as:

$$T\left(\bar{X}_{B1},\ldots,\bar{X}_{Bk};S_{B1}^2,\ldots,S_{Bk}^2\right)=\sum_{i=1}^{k}\frac{n_i}{S_{Bi}^2}\bar{X}_{Bi}^2-\frac{\left[\sum_{i=1}^{k}n_i\bar{X}_{Bi}\Big/S_{Bi}^2\right]^2}{\sum_{i=1}^{k}n_i\Big/S_{Bi}^2}, \qquad (6)$$

where $\bar{X}_{Bi}$ is distributed as $Z_i\left(S_i/\sqrt{n_i}\right)$ and Z_i is a standard normal random variable.

Then the PB test statistic in (6) is written as:

$$T_B\left(Z_i,\chi^2_{n_i-1};S_i^2\right)=\sum_{i=1}^{k}\frac{Z_i^2\left(n_i-1\right)}{\chi^2_{n_i-1}}-\frac{\left[\sum_{i=1}^{k}\left(\sqrt{n_i}Z_i\left(n_i-1\right)/S_i^2\chi^2_{n_i-1}\right)\right]^2}{\sum_{i=1}^{k}\left(n_i\left(n_i-1\right)/S_i^2\chi^2_{n_i-1}\right)}. \qquad (7)$$

The PB test rejects H_0 when:

$$P\left(T_B\left(Z_i,\chi^2_{n_i-1};S_i^2\right)>T_0\right)<\alpha, \tag{8}$$

where T_0 is the observed value of $T(\bar{X}_1, \ldots \bar{X}_k ; S_i^2, \ldots, S_k^2)$.

The original test, Welch's test, James's test and PB test are based on maximum likelihood estimators. In addition, there are some other interesting methods for estimation of parameters that can be used as an alternative to the aforementioned methods, one of which is Bayesian method. This method is based on the principle that the knowledge about parameter θ is assumed to be contained in a known prior distribution $\pi(\theta)$. Hence, updating distribution of parameter θ is the posterior distribution $\pi(\theta| x_1, \ldots, x_n)$, obtained by Bayes's theorem. When prior distribution is unknown, the Jeffrey's prior is derived from the sample distribution $f(x_1, \ldots, x_n| \theta)$, which can be used as an alternative to non-information prior distribution (Robert 2007).

In this study, we improve the test statistic by estimating population variance and population mean using Bayesian approach. The paper is organized as follows. The test statistic for ANOVA under Bayesian approach is described in the next section. After that, we shows comparisons of the performance made between and four existing test (the original test, James's test, Welch's test and the PB test) based on Type I error rates and power of the test. Finally, last section contains conclusion.

A TEST STATISTIC UNDER BAYESIAN APPROACH

Let X_{ij} be random variable from $N(\mu_i,\sigma_i^2)$, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n_i$. The probability density function is:

$$f(x|\mu_i,\sigma_i^2)=(2\pi)^{-\frac{1}{2}}(\sigma_i^2)^{-\frac{1}{2}}\exp\left\{-\frac{(x-\mu_i)^2}{2\sigma_i^2}\right\}. \tag{9}$$

Then, the likelihood function in the ith population is given by:

$$f(x_{i1},\ldots,x_{in_i}|\mu_i,\sigma_i^2)=(2\pi)^{-\frac{n_i}{2}}(\sigma_i^2)^{-\frac{n_i}{2}}\exp\left\{-\sum_{i=1}^{n_i}(x_{ij}-\mu_i)^2\Big/2\sigma_i^2\right\}. \tag{10}$$

From (10), we obtain the log likelihood function as

$$\ln f(x_{i1},\ldots,x_{in_i}|\mu_i,\sigma_i^2)=-\frac{n_i}{2}\ln(2\pi)-\frac{n_i}{2}\ln(\sigma_i^2)-\ln\sum_{j=1}^{n_i}(x_{ij}-\mu_i)^2\Big/2\sigma_i^2. \tag{11}$$

The Fisher information matrix for $\theta=\left(\mu_i,\sigma_i^2\right)$ is given as,

$$I(\theta)=-E\begin{bmatrix}\dfrac{\partial^2\ln f(X_{i1},\ldots,X_{in_i}|\mu_i,\sigma_i^2)}{\partial\mu_i^2} & \dfrac{\partial^2\ln f(X_{i1},\ldots,X_{in_i}|\mu_i,\sigma_i^2)}{\partial\mu_i\partial\sigma_i^2}\\ \dfrac{\partial^2\ln f(X_{i1},\ldots,X_{in_i}|\mu_i,\sigma_i^2)}{\partial\mu_i\partial\sigma_i^2} & \dfrac{\partial^2\ln f(X_{i1},\ldots,X_{in_i}|\mu_i,\sigma_i^2)}{\partial\left(\sigma_i^2\right)^2}\end{bmatrix}$$

$$=-E\begin{bmatrix}-\dfrac{n_i}{\sigma_i^2} & -\dfrac{\sum_{j=1}^{n_i}\left(X_{ij}-\mu_i\right)}{\left(\sigma_i^2\right)^2}\\ -\dfrac{\sum_{j=1}^{n_i}\left(X_{ij}-\mu_i\right)}{\left(\sigma_i^2\right)^2} & \dfrac{n_i}{2\left(\sigma_i^2\right)^2}-\dfrac{\sum_{j=1}^{n_i}\left(X_{ij}-\mu_i\right)^2}{\left(\sigma_i^2\right)^3}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{n_i}{\sigma_i^2} & 0\\ 0 & \dfrac{n_i}{2\left(\sigma_i^2\right)^2}\end{bmatrix}. \tag{12}$$

Therefore, the Jeffreys prior distribution for μ_i and σ_i^2 is:

$$\pi(\theta)\propto\left[\det\left(I(\theta)\right)\right]^{1/2}=\sqrt{\frac{n_i^2}{2\left(\sigma_i^2\right)^3}}=\frac{n_i}{2^{1/2}\left(\sigma_i^2\right)^{3/2}}\propto\left(\sigma_i^2\right)^{-\frac{3}{2}}. \tag{13}$$

Hence, a joint posterior distribution for μ_i and σ_i^2 is:

$$\pi(\mu_i,\sigma_i^2|x_{i1},\ldots,x_{in_i})=\frac{f(x_{i1},\ldots,x_{in_i}|\mu_i,\sigma_i^2)\left(\sigma_i^2\right)^{-\frac{3}{2}}}{\iint f(x_{i1},\ldots,x_{in_i}|\mu_i,\sigma_i^2)\left(\sigma_i^2\right)^{-\frac{3}{2}}d\mu_i\,d\sigma_i^2}$$

$$=\frac{(\sigma_i^2)^{-\frac{1}{2}(n_i+3)}\exp\left\{-\frac{1}{2\sigma_i^2}\left[n_i\left(\mu_i-\bar{x}_i\right)^2+\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2\right]\right\}}{\left(\frac{2\pi}{n_i}\right)^{1/2}\beta^{\alpha_1}\Gamma\left(\alpha_1\right)}, \tag{14}$$

where $\beta=\frac{2}{\sum_{j=1}^{n_i}\left(x_{ij}-\bar{x}_i\right)^2}$ and $\alpha_1=\frac{n_i}{2}$.

Thus, the posterior distribution for σ_i^2 could be:

$$\pi(\sigma_i^2 \big| x_{i1},...,x_{in_i}) = \int_{-\infty}^{\infty} \pi(\mu_i,\sigma_i^2 \big| x_{i1},...,x_{in_i}) d\mu_i$$

$$= \frac{(\sigma_i^2)^{-\frac{1}{2}(n_i+2)} \exp\left\{ -\frac{1}{2\sigma_i^2} \sum_{j=1}^{n_i} (x_{ij}-\bar{x}_i)^2 \right\}}{\beta^{\alpha_1}\Gamma(\alpha_1)}. \quad (15)$$

So, we obtain Bayes estimator of population variance by finding the expectation of σ_i^2 under the posterior distribution of σ_i^2 for given data,

$$E(\sigma_i^2 \big| x_{i1},...,x_{in_i}) = \int_0^{\infty} \sigma_i^2 \pi(\sigma_i^2 \big| x_{i1},...,x_{in_i}) d\sigma_i^2$$

$$= \frac{\sum_{j=1}^{n_i} (x_{ij}-\bar{x}_i)^2}{n_i - 2} = \hat{\sigma}^2_{Bayes_i} \quad (16)$$

Furthermore, we also estimate population mean by using Bayesian method, where the posterior distribution for μ_i is:

$$\pi(\mu_i \big| x_{i1},...,x_{in_i}) = \frac{(\sigma_i^2)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2\sigma_i^2}\left[n_i\left(\mu_i - \bar{x}_i\right)^2 \right] \right\}}{\left(\frac{2\pi}{n_i}\right)^{1/2}} \quad (17)$$

So, Bayes estimator of population mean obtained by finding the expectation of μ_i under the posterior distribution of μ_i for given can be written as follows:

$$E(\mu_i \big| x_{i1},...,x_{in_i}) = \int_0^{\infty} \mu_i \pi(\mu_i \big| x_{i1},...,x_{in_i}) d\mu_i$$

$$= \frac{\sum_{j=1}^{n_i} x_{ij}}{n_i} = \bar{x}_{Bayes_i} \quad (18)$$

After that, the parameters in (1) are replaced by the Bayes estimators of population mean and population variance and the statistic using Bayes estimator (say, the proposed test) can be written as:

$$T\left(\bar{X}_{Bayes_1},...,\bar{X}_{Bayes_k};\hat{\sigma}^2_{Bayes_1},...,\hat{\sigma}^2_{Bayes_k}\right) = \sum_{i=1}^{k} \frac{n_i}{\hat{\sigma}^2_{Bayes_i}}$$

$$\bar{X}^2_{Bayes_i} - \frac{\left[\sum_{i=1}^{k} n_i \bar{X}_{Bayes_i} \Big/ \hat{\sigma}^2_{Bayes_i} \right]^2}{\sum_{i=1}^{k} n_i \Big/ \hat{\sigma}^2_{Bayes_i}} \quad (19)$$

In the next section, the performance of the test statistics in ANOVA model using Type I error rates and power of the test is assessed.

TYPE I ERROR RATES AND POWER OF THE TEST

In this study, we perform a Monte Carlo simulation consisting of 10000 iterations to compute the Type I error rates and power of the test among the original test, James's test, Welch's test, the PB test and the proposed test. As previously mentioned, we suppose, without loss of generality, that $\mu_1 = \ldots = \mu_k = 0$, $\sigma_i^2 = 1$ and $0 < \sigma_i^2 < 1$, $i = 2, \ldots k$ for $k = 3, 6$ and 10, under various values of n_i in our simulation studies. Therefore, the sample statistics $\bar{x}_i$ and s_i^2 will be generated independently as $\bar{x}_i \sim N\left(0, \sigma_i^2 / n_i\right)$ and $s_i^2 \sim \sigma_i^2 \chi^2_{n_i} / \left(n_i - 1\right)$. For the procedure of PB test, the Type I error rates of the PB test are computed by the proportion of the 10000 p-values that are less than the nominal level α (0.05). Power of the tests is also considered in this study. We perform a Monte Carlo simulation that consists of 10000 iterations so as to compute power of the tests for the original test, James's test, Welch's test, the PB test and the proposed test. We assume the conditions of population means for k = 3 with $\mu_1 = 0$ and with μ_2 and μ_3 having different values and for k = 10 with $\mu_1, \ldots, \mu_8$ = 0 and with μ_8 and μ_9 having different levels where the population variances are $0 < \sigma_i^2 < 1$, $i = 2, \ldots, k$ under selected values of n_i.

The results of Type I error rates and power of the test of the original test, James's test, Welch's test, the PB test and the proposed test are presented in Tables 1 and 2, respectively.

Table 1 shows the Type I error rates for nominal level 0.05 when k = 3, 6 and 10 and when sample sizes are equal and unequal for some selected values of variances. The results of Table 1 indicates that for k = 3 and when n = (5, 5, 5), n = (10, 10, 10), n = (4, 6, 20), n = (3, 4, 3) and for k = 6 and when n = (5,5,5,5,5,5), n = (10,10,10,10,10), n = (3, 3. 4, 5, 6, 6), n = (4, 8, 12, 24, 30, 40) in case variances are equal and unequal, James's test, Welch's test, the PB test and the proposed test perform well and have similar Type I error rates. The original test seems to poorly control Type I error rates for almost all cases.

For k = 10 and when n = (5,5,5,5,5,5,5,5,5,5), n = (15,15,15,15,15,15,15,15,15,15) and variances are equal,

TABLE 1. Type I error rates for $\alpha = 0.05$

$k = 3, \sigma_1^2 = 1$	$n = (5, 5, 5)$					$n = (10, 10, 10)$					$n = (4, 6, 20)$					$n = (3, 4, 3)$				
(σ_2^2, σ_3^2)	P	O	W	J	PB	P	O	W	J	PB	P	O	W	J	PB	P	O	W	J	PB
(1, 1)	0.05	0.08	0.04	0.05	0.05	0.04	0.05	0.03	0.04	0.05	0.05	0.12	0.05	0.06	0.05	0.04	0.09	0.04	0.06	0.05
(1, 0.5)	0.05	0.12	0.04	0.05	0.05	0.05	0.07	0.04	0.04	0.05	0.06	0.14	0.04	0.09	0.05	0.05	0.13	0.06	0.07	0.05
(1, 0.1)	0.05	0.11	0.05	0.06	0.05	0.03	0.04	0.03	0.03	0.05	0.06	0.11	0.03	0.07	0.06	0.05	0.16	0.03	0.08	0.06
(0.5, 0.5)	0.04	0.08	0.03	0.05	0.05	0.05	0.07	0.03	0.04	0.05	0.03	0.06	0.03	0.03	0.06	0.05	0.18	0.04	0.08	0.06
(0.5, 0.7)	0.04	0.11	0.04	0.06	0.05	0.06	0.08	0.05	0.06	0.05	0.05	0.09	0.03	0.05	0.06	0.06	0.15	0.03	0.08	0.05
(0.1, 0.1)	0.05	0.07	0.03	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.04	0.06	0.04	0.04	0.06	0.05	0.14	0.02	0.09	0.05
(0.1, 0.9)	0.04	0.08	0.04	0.05	0.04	0.03	0.05	0.02	0.03	0.05	0.05	0.14	0.05	0.08	0.06	0.06	0.15	0.05	0.09	0.05
(0.5, 0.9)	0.04	0.10	0.04	0.06	0.04	0.04	0.07	0.04	0.06	0.04	0.03	0.09	0.03	0.04	0.06	0.05	0.12	0.04	0.08	0.05
(0.3, 0.9)	0.05	0.10	0.04	0.06	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.09	0.04	0.07	0.05	0.04	0.12	0.04	0.08	0.05
(0.3, 0.6)	0.05	0.06	0.04	0.06	0.05	0.05	0.06	0.03	0.05	0.05	0.05	0.10	0.04	0.07	0.05	0.05	0.15	0.04	0.09	0.06
(0.1, 0.3)	0.05	0.10	0.04	0.06	0.05	0.03	0.06	0.03	0.03	0.05	0.06	0.11	0.05	0.08	0.05	0.06	0.14	0.05	0.09	0.06
(0.05, 0.05)	0.04	0.09	0.04	0.05	0.05	0.04	0.05	0.03	0.04	0.05	0.05	0.10	0.05	0.07	0.06	0.05	0.15	0.04	0.10	0.05
$k = 6, \sigma_1^2 = 1$	$n = (5,5,5,5,5,5)$					$n = (10,10,10,10,10)$					$n = (3, 3. 4, 5, 6, 6)$					$n = (4, 8, 12, 24, 30, 40)$				
(σ_2^2, σ_6^2)	P	O	W	J	PB	P	O	W	J	PB	P	O	W	J	PB	P	O	W	J	PB
(1, 1, 1, 1, 1)	0.05	0.09	0.05	0.06	0.05	0.04	0.05	0.04	0.04	0.05	0.05	0.18	0.06	0.06	0.05	0.05	0.07	0.04	0.06	0.05
(0.1, 0.1, 0.5, 0.5, 0.5)	0.05	0.14	0.08	0.07	0.05	0.05	0.06	0.05	0.05	0.05	0.06	0.15	0.06	0.06	0.06	0.05	0.08	0.05	0.06	0.05
(0.1, 0.2, 0.3, 0.4, 0.5)	0.06	0.18	0.08	0.07	0.05	0.05	0.10	0.05	0.05	0.05	0.06	0.15	0.06	0.06	0.06	0.05	0.12	0.05	0.05	0.04
(0.1, 1, 1, 1, 1)	0.06	0.19	0.08	0.06	0.05	0.05	0.06	0.05	0.05	0.05	0.06	0.19	0.05	0.06	0.05	0.05	0.05	0.05	0.05	0.05
(0.2, 0.4, 0.4, 0.2, 0.1)	0.06	0.15	0.07	0.06	0.05	0.05	0.06	0.05	0.05	0.05	0.06	0.21	0.05	0.06	0.05	0.05	0.12	0.05	0.05	0.05
(0.5, 0.5, 0.5, 0.5, 1)	0.06	0.16	0.08	0.07	0.05	0.05	0.07	0.05	0.05	0.05	0.06	0.20	0.05	0.06	0.06	0.05	0.13	0.05	0.05	0.04
(0.3, 0.9, 0.4, 0.7, 0.1)	0.06	0.20	0.09	0.07	0.05	0.05	0.06	0.05	0.05	0.05	0.06	0.23	0.05	0.06	0.06	0.06	0.14	0.05	0.05	0.05
(0.01, 0.01, 0.06, 0.1, 0.1)	0.06	0.09	0.05	0.06	0.05	0.05	0.06	0.05	0.05	0.05	0.06	0.21	0.05	0.06	0.06	0.05	0.12	0.05	0.05	0.05
$k = 10, \sigma_1^2 = 1$	$n = (5,5,5,5,5,5,5,5,5,5)$					$n = (15,15,15,15,15,15,15,15,15,15)$					$n = (3,3,3, 4, 4, 5, 5, 5, 5)$					$n = (4, 4, 4, 12, 12, 15, 15, 15, 15)$				
$(\sigma_2^2, \sigma_{10}^2)$	P	O	W	J	PB	P	O	W	J	PB	P	O	W	J	PB	P	O	W	J	PB
(1, 1, 1, 1, 1, 1, 1, 1, 1)	0.05	0.09	0.05	0.06	0.05	0.03	0.05	0.03	0.05	0.05	0.07	0.021	0.08	0.14	0.05	0.08	0.19	0.06	0.12	0.05
(0.1,0.2,0.3,0.4,0.5,0.6,0.7, 0.8, 0.9)	0.06	0.14	0.08	0.07	0.05	0.05	0.06	0.05	0.05	0.05	0.14	0.21	0.10	0.15	0.05	0.06	0.17	0.05	0.09	0.05
(0.1,0.1,0.2,0.2,0.3,0.3, 0.4, 0.4,0.5)	0.06	0.18	0.08	0.07	0.05	0.06	0.10	0.06	0.07	0.05	0.09	0.19	0.10	0.09	0.04	0.08	0.014	0.6	0.08	0.05
(0.1,0.1,0.1,0.1,0.1,0.2,0.2, 0.2,0.2)	0.06	0.19	0.08	0.06	0.05	0.05	0.15	0.06	0.06	0.05	0.09	0.19	0.08	0.08	0.05	0.06	0.10	0.08	0.07	0.05
(0.1,1,0.1,1,0.1,1,0.1,1, 0.1)	0.05	0.15	0.07	0.06	0.05	0.05	0.10	0.06	0.06	0.05	0.08	0.14	0.08	0.09	0.04	0.07	0.15	0.07	0.07	0.05
(0.3,0.3,0.3,0.6,0.6,0.6, 0.9, 0.9,0.9)	0.06	0.16	0.08	0.07	0.05	0.06	0.12	0.06	0.06	0.05	0.07	0.18	0.09	0.07	0.05	0.06	0.11	0.07	0.07	0.05
(0.1,0.1,0.1,0.1,0.1, 0.1, 0.1, 0.1,0.1)	0.06	0.20	0.09	0.07	0.04	0.05	0.18	0.05	0.06	0.05	0.07	0.19	0.07	0.07	0.05	0.07	0.09	0.07	0.06	0.05

P-The proposed test; O-The original test; W-Welch's test; J-James's test; PB-The parametric bootstrap test

TABLE 2. Power of the tests

$k = 3, \sigma_1^2 = 1$ and $\mu_1 = 0$		(μ_2, μ_3)						
$n = (10, 10, 10)$								
(σ_2^2, σ_3^2)	Tests	(0, 0)	(0, 0.2)	(0, 0.5)	(0, 0.7)	(0.5, 1)	(0, 1)	(1.5, 1)
(0.3, 0.9)	P	0.05	0.09	0.24	0.44	0.51	0.72	0.95
	O	0.06	0.08	0.25	0.45	0.52	0.74	0.95
	W	0.05	0.07	0.21	0.40	0.49	0.67	0.93
	J	0.05	0.08	0.23	0.42	0.49	0.71	0.94
	PB	0.05	0.08	0.22	0.40	0.49	0.67	0.94
(0.1, 0.5)	P	0.05	0.12	0.42	0.69	0.65	0.95	0.98
	O	0.07	0.14	0.43	0.69	0.66	0.95	0.98
	W	0.05	0.09	0.38	0.63	0.60	0.91	0.97
	J	0.06	0.11	0.39	0.67	0.63	0.93	0.98
	PB	0.05	0.10	0.39	0.64	0.61	0.92	0.98
$n = (10, 5, 15)$								
(0.3, 0.9)	P	0.05	0.09	0.28	0.49	0.60	0.80	0.89
	O	0.07	0.10	0.30	0.52	0.63	0.82	0.90
	W	0.05	0.08	0.23	0.45	0.52	0.75	0.87
	J	0.06	0.09	0.27	0.48	0.57	0.78	0.89
	PB	0.05	0.09	0.24	0.46	0.55	0.79	0.89
(0.1, 0.5)	P	0.05	0.13	0.49	0.77	0.75	0.98	0.98
	O	0.07	0.15	0.50	0.79	0.77	0.97	0.97
	W	0.05	0.11	0.42	0.73	0.68	0.96	0.96
	J	0.05	0.13	0.47	0.76	0.70	0.97	0.97
	PB	0.05	0.11	0.42	0.73	0.68	0.96	0.96

(continue)

Continued TABEL 2

$k = 10$ and $(\mu_1, \ldots, \mu_8) = 0$ $n = (15, 15, 15, 20, 20, 20, 25, 25, 25, 25)$	Tests	(μ_9, μ_{10}) (0, 0)	(0, 0.2)	(0, 0.5)	(0, 0.7)	(0.5, 1)	(0, 1)	(1.5, 1)
$(\sigma_1^2,\ldots,\sigma_{10}^2) = (1, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$	P	0.05	0.11	0.43	0.71	0.99	0.96	1
	O	0.06	0.13	0.45	0.74	0.99	0.96	1
	W	0.05	0.09	0.36	0.63	0.99	0.93	1
	J	0.05	0.13	0.43	0.70	0.99	0.96	1
	PB	0.05	0.11	0.40	0.67	0.99	0.95	1
$(\sigma_1^2,\ldots,\sigma_{10}^2) = (1, 0.1, 0.1, 0.1, 0.3, 0.3, 0.3, 0.7, 0.7, 0.7)$	P	0.05	0.15	0.48	0.81	0.99	0.99	1
	O	0.06	0.16	0.50	0.81	0.99	0.99	1
	W	0.05	0.09	0.43	0.77	0.99	0.98	1
	J	0.05	0.10	0.48	0.80	0.99	0.99	1
	PB	0.05	0.14	0.46	0.80	0.99	0.99	1
$n = (15,17,19,21,23,25,27,29,31,33)$								
$(\sigma_1^2,\ldots,\sigma_{10}^2) = (1,0.1,0.2,0.3,0.4, 0.5,0.6,0.7,0.8,0.9)$	P	0.05	0.13	0.53	0.85	0.99	0.99	1
	O	0.06	0.15	0.55	0.87	0.99	0.99	1
	W	0.05	0.10	0.46	0.81	0.99	0.98	1
	J	0.05	0.13	0.54	0.86	0.99	0.99	1
	PB	0.05	0.12	0.48	0.81	0.99	0.99	1
$(\sigma_1^2,\ldots,\sigma_{10}^2) = (1, 0.1, 0.1, 0.1, 0.3, 0.3, 0.3, 0.7, 0.7, 0.7)$	P	0.05	0.15	0.63	0.92	1	0.99	1
	O	0.06	0.17	0.66	0.92	1	0.99	1
	W	0.05	0.12	0.56	0.89	1	0.99	1
	J	0.05	0.17	0.65	0.92	1	0.99	1
	PB	0.05	0.14	0.59	0.90	1	0.99	1

P-The proposed test; O-The original test; W-Welch's test; J-James's test; PB-The parametric bootstrap test

James's test, Welch's test, the PB test and the proposed test appear to control Type I error rates quite well. Meanwhile, the PB test tends to control Type I error rates very well as observed for unequal sample sizes with n = (3, 3, 3, 4, 4, 5, 5, 5, 5) and n = (4, 4, 4, 12, 12, 15, 15, 15, 15) for equal and unequal variances.

We again note that in any cases, the proposed test can control Type I error rates better than the original test.

Table 2 shows power of the tests of the five tests. We observe that for k = 3, the proposed test and the original test appear to have high power of the test for sample sizes n = (10, 10, 10) and n = (10, 5, 15). When k = 10 and n = (15, 15, 15, 20, 20, 20, 25, 25, 25, 25) all the tests appear to have high power of the test with Welch's test showing less power of the test than the other four tests. For n = (15, 17, 19, 21, 23, 25, 27, 29, 31, 33), the proposed test and the original test seem to have high power of the test.

Consequently, we observe that the proposed test and the original test seem to have high power of the test with their values higher than those observed for James's test, Welch's test and the PB test. However, based on the results of Type I error rate, the proposed test has preference over the original test.

Conclusion

The original test for one-way ANOVA with unequal variances has a serious Type I error problem and some tests like James's test, Welch's test and the PB test have been proposed to solve the problem. However, these tests show difficulty in calculation. According to this, the test for ANOVA with estimation of parameters under Bayesian approach is proposed as an alternative to the aforementioned tests. The proposed test is compared with the original test, James's test, Welch's test, and the PB test in terms of Type I error rate and power of the test, and it is observed that for k = 3 and 6, the proposed test controls Type I error rate quite well, which is comparable to James's test, Welch's test and the PB test. By considering power of the test, high power of the test is observed for all the tests including the original test. Meanwhile, for k = 10, James's test, Welch's test, the PB test and the proposed test appear to control the Type I error rate well when sample sizes are equal for each group, which is well-supported by Box (1951). However, the PB test seems to control Type I error rate better than the other tests when sample sizes are unequal. Results on power of the test show that the proposed test seems to have high power of the test compared with the other tests. Our results suggest that the proposed test has high performance comparable to James's test, Welch's test and the PB test and should be used as an alternative approach because of its simple formula.

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