A Sixth-Order RKFD Method with Four-Stage for Directly Solving Special Fourth-Order ODEs

(Kaedah RKFD Peringkat Keenam dengan Tahap Empat untuk Menyelesaikan Secara Terus PPB Khas Peringkat Keempat)

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ABSTRACT

In this article, the general form of Runge-Kutta method for directly solving a special fourth- order ordinary differential equations denoted as RKFD method is given. The order conditions up to order seven are derived, based on the order conditions, we construct a new explicit four-stage sixth-order RKFD method denoted as RKFD6 method. Zero-stability of the method is proven. Comparisons are made using the existing Runge–Kutta methods after the problems are reduced to a system of first order ordinary differential equations. Numerical results are presented to illustrate the efficiency and competency of the new method.

Keywords: Ordinary differential equations; special fourth order; RKFD method; Runge-Kutta method

ABSTRAK

Dalam kertas ini, bentuk umum kaedah Runge-Kutta untuk menyelesaikan secara terus persamaan pembezaan biasa khas peringkat keempat yang ditandakan sebagai kaedah RKFD diberikan. Syarat tertib hingga ke peringkat ketujuh diterbitkan, berasaskan syarat ini, kami bina kaedah baharu RKFD tahap empat peringkat keenam yang ditandakan sebagai RKFD6. Kestabilan sifar kaedah ini dibuktikan. Perbandingan dijalankan menggunakan kaedah Runge-Kutta sedia ada setelah masalah tersebut diturunkan kepada sistem persamaan pembezaan peringkat pertama. Keputusan berangka dipersembahkan untuk menunjukkan kecekapan dan kompetensi kaedah yang baharu tersebut.

Kata kunci: Kaedah RKFD; kaedah Runge-Kutta; peringkat keempat khas; persamaan pembezaan biasa

INTRODUCTION

In this paper, we are concerned with the efficiency of the numerical integration of the special fourth-order ordinary differential equations (ODEs) of the form

$$y^{(iv)}(x) = f(x,y), \qquad (1)$$

with initial conditions

$$y(x_0) = y_0, \, y'(x_0) = y'_0, \, y''(x_0) = y''_0, y'''(x_0) = y'''_0,$$

where $f : R \times R^m \to R^m$ is a continuous valued function which does not contain the first, second and third derivatives. The fourth-order ordinary differential equations include initial or boundary value problems, which arise in various of applied fields such as quantum mechanics, electronics, molecular dynamics and control engineering. Traditionally, researchers and engineers solve the fourth-order ODEs by converting them into a first-order system of ODEs and then applying a suitable numerical methods to solve the resulting system (Butcher 2008; Dahlquist 1978; Hairer et al. 2010; Lambert 1991; Onumanyi et al. 1999). However, the application of such techniques takes a lot of computational time. Direct integration method is proposed to avoid such computational burden and increase

the efficiency of the method. Many authors have proposed several numerical methods for directly approximating the solutions for the higher order ODEs, for example Kayode (2008) proposed a zero stable predictor-corrector methods for solving fourth-order ordinary differential equations. Majid and Suleiman (2006) derived one point block method to solve system of higher order ODEs. A six point block method for solving fourth order ODEs without using the predictor-corrector methods derived by Olabode (2009). Waeleh et al. (2011) constructed a new block method for solving directly higher order ODEs. Awoyemi and Idowu (2005) proposed a hybrid collocations method for solving third-order ODEs. Hybrid linear multistep method with three steps to solve second-order ODEs was introduced by Jator (2011). All these methods are multistep and are not self-starting. Therefore, these methods require the starting method to obtain the initial values for solving (1).

This paper primarily aimed to construct a one-step method of order six to solve special fourth-order ODEs directly. This new method is self-starting in nature. The paper is organized as follows: In the next section, we give the general form of RKFD method. The order conditions up to seventh order for RKFD method are derived in the following section. Next, the zero-stability of RKFD method is presented. In section that follows, a new explicit four-

stage six-order RKFD method is constructed, namely RKFD6 method. Numerical experiments are given after that to show the effectiveness and competence of the new RKFD6 method as compared with the well known Runge-Kutta (RK) methods from the scientific literature. Conclusions is given in the last section.

THE GENERAL FORM OF THE RKFD METHOD

In this section we present the general form of RKFD method with -stage for directly solving special fourth-order ODEs (1) as follows

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n + h^4\sum_{i=1}^{s}b_ik_i, \tag{2}$$

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{2}y'''_n + h^3\sum_{i=1}^{s}b'_ik_i, \tag{3}$$

$$y''_{n+1} = y''_n + hy'''_n + h^2\sum_{i=1}^{s}b''_ik_i, \tag{4}$$

$$y'''_{n+1} = y'''_n + h\sum_{i=1}^{s}b'''_ik_i, \tag{5}$$

where

$$k_1 = f(x_n, y_n),$$

$$k_i = f(x_n + c_ih, y_n + hc_iy'_n + \frac{h^2}{2}c_i^2y''_n + \frac{h^3}{6}c_i^3y'''_n + h^4\sum_{j=1}^{i-1}a_{ij}k_j),$$

$$i = 2, 3, \ldots, s. \tag{6}$$

All parameters $b_i, b'_i, b''_i, b'''_i, a_{ij}$ and c_i of the RKFD method are used for $i = 1, 2, \ldots, s; j = 1, 2, \ldots, s$ and suppose to be real. The RKFD method is an explicit method if $a_{ij} = 0$ for $i \le j$ and is an implicit method if $a_{ij} \ne 0$ for $i \le j$. The scheme (2)-(6) can be expressed in Butcher tableau as follows:

$$
\begin{array}{c|c}
c & A \\
\hline
 & b^T \\
 & b'^T \\
 & b''^T \\
 & b'''^T
\end{array}
$$

In order to determine the parameters of the RKFD method given by (2)-(6), the RKFD method expressions (2)-(6) are expanded using the Taylor series expansion. After performing some algebraic manipulations, this expansion is equated to the true solution that is given by the Taylor series expansion. The direct expansion of the truncation error is used to derive the order conditions for the RKFD method. A good deal of algebraic and numerical calculations which is required for the above operation which were carried out using algebra package MAPLE.

ORDER CONDITIONS OF THE RKFD METHOD

Hussain et al. (2015) derived the algebraic order conditions for the RKFD method up to order five. Hence we used the same technique which is the Taylor series expansion to obtain the order conditions for RKFD method (2)-(6) up to order seven, which can be written as follows:
The order conditions for y:

order 4: $\sum_{i=1}^{s}b_i = \frac{1}{24},$ (7)

order 5: $\sum_{i=1}^{s}b_ic_i = \frac{1}{120},$ (8)

order 6: $\sum_{i=1}^{s}b_ic_i^2 = \frac{1}{360},$ (9)

order 7: $\sum_{i=1}^{s}b_ic_i^3 = \frac{1}{840},$ (10)

The order conditions for y':

order 3: $\sum_{i=1}^{s}b'_i = \frac{1}{6},$ (11)

order 4: $\sum_{i=1}^{s}b'_ic_i = \frac{1}{24},$ (12)

order 5: $\sum_{i=1}^{s}b'_ic_i^2 = \frac{1}{60},$ (13)

order 6: $\sum_{i=1}^{s}b'_ic_i^3 = \frac{1}{120},$ (14)

order 7: $\sum_{i=1}^{s}b'_ic_i^4 = \frac{1}{210},$ (15)

$$\sum_{i,j=1}^{s}b'_ia_{ij} = \frac{1}{5040}, \tag{16}$$

The order conditions for y'':

order 2: $\sum_{i=1}^{s}b''_i = \frac{1}{2},$ (17)

order 3: $\sum_{i=1}^{s}b''_ic_i = \frac{1}{6},$ (18)

order 4: $\sum_{i=1}^{s}b''_ic_i^2 = \frac{1}{12},$ (19)

order 5: $\sum_{i=1}^{s}b''_ic_i^3 = \frac{1}{20},$ (20)

order 6: $\sum_{i=1}^{s}b''_ic_i^4 = \frac{1}{30},$ (21)

$$\sum_{i,j=1}^{s}b''_ia_{ij} = \frac{1}{720}, \tag{22}$$

order 7: $\sum_{i=1}^{s}b''_ic_i^5 = \frac{1}{42},$ (23)

$$\sum_{i,j=1}^{s}b''_ia_{ij}c_j = \frac{1}{5040}, \quad \sum_{i,j=1}^{s}b''_ic_ia_{ij} = \frac{1}{1008}, \tag{24}$$

The order conditions for y''':

order 1: $\sum_{i=1}^{s}b'''_i = 1,$ (25)

order 2: $\sum_{i=1}^{s}b'''_ic_i = \frac{1}{2},$ (26)

order 3: $\sum_{i=1}^{s}b'''_ic_i^2 = \frac{1}{3},$ (27)

order 4: $\sum_{i=1}^{s}b'''_ic_i^3 = \frac{1}{4},$ (28)

order 5: $\sum_{i=1}^{s}b'''_ic_i^4 = \frac{1}{5},$ (29)

$$\sum_{i,j=1}^{s} b_i''' a_{ij} = \tfrac{1}{120}, \tag{30}$$

order 6: $\sum_{i=1}^{s} b_i''' c_i^5 = \tfrac{1}{6},$ (31)

$$\sum_{i,j=1}^{s} b_i''' a_{ij} c_j = \tfrac{1}{720}, \quad \sum_{i,j=1}^{s} b_i''' c_i a_{ij} = \tfrac{1}{144}, \tag{32}$$

order 7: $\sum_{i=1}^{s} b_i''' c_i^6 = \tfrac{1}{7},$ (33)

$$\sum_{i,j=1}^{s} b_i''' c_i a_{ij} c_j = \tfrac{1}{840}, \ \sum_{i,j=1}^{s} b_i''' a_{ij} c_j^2 = \tfrac{1}{2520}, \tag{34}$$

$$\sum_{i,j=1}^{s} b_i''' c_i^2 a_{ij} = \tfrac{1}{168}, \tag{35}$$

ZERO-STABILITY OF THE RKFD METHOD

In this section, we investigate the convergence of the RKFD method by introducing the concept of zero-stability of the RKFD method. A good numerical method is a method in which the numerical approximation to the solution converges and zero-stability is a significant criterion for convergence. The zero-stability concept for those numerical methods that are used for solving first and second order ODEs can be seen in Butcher (2008), Dormand (1996) and Lambert (1991). The RKFD method (2)-(6) can be expressed in the matrix form as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ hy'_{n+1} \\ h^2 y''_{n+1} \\ h^3 y'''_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \tfrac{1}{2} & \tfrac{1}{6} \\ 0 & 1 & 1 & \tfrac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \\ hy'_n \\ h^2 y''_n \\ h^3 y'''_n \end{bmatrix},$$
(36)

where $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

is the identity matrix coefficients of $y_{n+1}, hy'_{n+1}, h^2 y''_{n+1}$, and $h^3 y'''_{n+1}$, respectively,

and $A = \begin{bmatrix} 1 & 1 & \tfrac{1}{2} & \tfrac{1}{6} \\ 0 & 1 & 1 & \tfrac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

is a matrix coefficients of $y_n, hy'_n, h^2 y''_n$, and $h^3 y'''_n$, respectively.

The characteristic polynomial of the RKFD method is called $\varrho(\zeta)$ which can be written as follows:

$$\varrho(\zeta) = |I\zeta - A| = \begin{vmatrix} \zeta-1 & -1 & -\tfrac{1}{2} & -\tfrac{1}{6} \\ 0 & \zeta-1 & -1 & -\tfrac{1}{2} \\ 0 & 0 & \zeta-1 & -1 \\ 0 & 0 & 0 & \zeta-1 \end{vmatrix}. \tag{37}$$

Hence,

$$\varrho(\zeta) = (\zeta - 1)^4, \tag{38}$$

we find that all the roots are $\zeta = 1,1,1,1$. Generalizing the theorem proposed by Henrici (1962) for solving special second order ODEs, therefore, the RKFD method is zero-stable since no root of the polynomial exceeds 1 and that the multiplicity of the roots of modulus 1 is at most 4. It is well-known that for a method to be convergent it has to be consistent and zero-stable.

For the first order ODE $y'(x) = f(x, y)$, the method is consistent if its order is at least one. For the special second order ODE $y''(x) = f(x, y)$, (p. 301 in Henrici (1962), the numerical method is consistent if the order is at least two. So without loss of generality we can say that for the special third order ODE $y'''(x) = f(x, y)$, the method is consistent if the order is at least three and for the special fourth order ODE $y^{(iv)}(x) = f(x, y)$, the method is consistent if the order is at least four. Hence the method we are going to derive is sixth order therefore it is consistent.

SIXTH ORDER RKFD METHOD WITH FOUR-STAGE

In order to construct the four-stage sixth-order RKFD6 method the algebraic conditions of RKFD method up to order six for y, y', y'' and y''' need to be solved. We choose (11)-(14) from order conditions for y', (17)-(21) from order conditions for y'' and (25)-(29) and (31) from order conditions for y'''. Consequently, the system of equations consists of 15 non-linear equations with 15 unknowns need to be solved. This system has no free parameter but is consistent, solving the system simultaneously yields a a unique solution as follows:

$b_1' = \tfrac{1}{24}, b_2' = \tfrac{1}{16} - \tfrac{\sqrt{5}}{48}, b_3' = \tfrac{1}{16} + \tfrac{\sqrt{5}}{48}, b_4' = 0, b_1'' = \tfrac{1}{12}, b_2'' = \tfrac{5}{24} - \tfrac{\sqrt{5}}{24},$

$b_3'' = \tfrac{5}{24} + \tfrac{\sqrt{5}}{24}, b_4'' = 0, b_1''' = \tfrac{1}{12}, b_2''' = \tfrac{5}{12}, b_3''' = \tfrac{5}{12}, b_4''' = \tfrac{1}{12},$

$c_2 = \tfrac{1}{2} + \tfrac{\sqrt{5}}{10}, \ c_3 = \tfrac{1}{2} - \tfrac{\sqrt{5}}{10}, \ c_4 = 1.$

Substituting the values of c_2, c_3 and c_4 into order conditions for y we get

$b_2 = \tfrac{1}{72} - \tfrac{7\sqrt{5}}{360} + \sqrt{5}b_1, \ b_3 = \tfrac{1}{72} + \tfrac{7\sqrt{5}}{360} - \sqrt{5}b_1, \ b_4 = -b_1 + \tfrac{1}{72}.$

Here we have one free parameter b_1 which can be chosen by minimizing the error norm of the seventh order conditions for y according to Dormand et al. (1987). The

error norms and the global error norm of the seventh order conditions are defined as follows:

$$\left\|\tau^{(7)}\right\|_2 = \sqrt{\sum_{i=1}^{n_p+1}\left(\tau_i^{(7)}\right)^2}, \qquad \left\|\tau'^{(7)}\right\|_2 = \sqrt{\sum_{i=1}^{n_p'+1}\left(\tau_i'^{(7)}\right)^2}, \quad (39)$$

$$\left\|\tau''^{(7)}\right\|_2 = \sqrt{\sum_{i=1}^{n_p''+1}\left(\tau_i''^{(7)}\right)^2}, \qquad \left\|\tau'''^{(7)}\right\|_2 = \sqrt{\sum_{i=1}^{n_p'''+1}\left(\tau_i'''^{(7)}\right)^2}, \quad (40)$$

$$\left\|\tau_g^{(7)}\right\|_2 = \sqrt{\sum_{i=1}^{n_p+1}\left(\tau_i^{(7)}\right)^2 + \sum_{i=1}^{n_p'+1}\left(\tau_i'^{(7)}\right)^2 + \sum_{i=1}^{n_p''+1}\left(\tau_i''^{(7)}\right)^2 + \sum_{i=1}^{n_p'''+1}\left(\tau_i'''^{(7)}\right)^2}, \quad (41)$$

where $\tau^{(7)}, \tau'^{(7)}, \tau''^{(7)}$ and $\tau'''^{(7)}$ are the local truncation error norms for y, y', y'' and y''' of the RKFD method, respectively and $\tau_g^{(7)}$ is the global error norm. As a result we find the error equation of y as follows:

$$\left\|\tau^{(7)}\right\|_2 = \tfrac{1}{6300}\sqrt{\left(-17 + 1260\, b_1\right)^2}. \quad (42)$$

By minimizing $\left\|\tau^{(7)}\right\|_2$ with respect to the free parameter b_1, we obtain $b_1 = \frac{17}{1260}$ is the optimal value which produces:

$$\left\|\tau^{(7)}\right\|_2 = 0, b_2 = \frac{1}{72} - \frac{\sqrt{5}}{168}, b_3 = \frac{1}{72} + \frac{\sqrt{5}}{168}, b_4 = \frac{1}{2520}.$$

In order to obtain the values of coefficients and $a_{21}, a_{31}, a_{32}, a_{41}, a_{42}$ and a_{43}. Substituting the values of $b_1, b_2, b_3, b_4, b_1', b_2', b_3', b_4', b_1'', b_2'', b_3'', b_4'', b_1''', b_2''', b_3'''$ and b_4''' into (30), (32) and (34) of the order conditions of y''' and (16) of the order conditions of y', which yields the following system of non-linear equations, which needs to be solved

$$\tfrac{1}{12}a_{32} + \left(\tfrac{1}{24} + \tfrac{\sqrt{5}}{120}\right)a_{42} + \left(\tfrac{1}{24} - \tfrac{\sqrt{5}}{120}\right)a_{43} = \tfrac{1}{840}, \quad (43)$$

$$\left(\tfrac{1}{8} + \tfrac{\sqrt{5}}{24}\right)a_{32} + \left(\tfrac{1}{40} + \tfrac{\sqrt{5}}{120}\right)a_{42} + \left(\tfrac{1}{40} - \tfrac{\sqrt{5}}{120}\right)a_{43} = \tfrac{1}{2520}, \quad (44)$$

$$\left(\tfrac{1}{16} - \tfrac{\sqrt{5}}{48}\right)a_{21} + \left(\tfrac{1}{16} + \tfrac{\sqrt{5}}{48}\right)\left(a_{31} + a_{32}\right) = \tfrac{1}{5040}, \quad (45)$$

$$\tfrac{5}{12}\left(a_{21} + a_{31} + a_{32}\right) + \tfrac{1}{12}\left(a_{41} + a_{42} + a_{43}\right) = \tfrac{1}{120}, \quad (46)$$

$$\left(\tfrac{5}{24} + \tfrac{5\sqrt{5}}{120}\right)a_{32} + \left(\tfrac{1}{24} + \tfrac{\sqrt{5}}{120}\right)a_{42} + \left(\tfrac{1}{24} - \tfrac{\sqrt{5}}{120}\right)a_{43} = \tfrac{1}{720}, \quad (47)$$

$$\left(\tfrac{5}{24} + \tfrac{5\sqrt{5}}{120}\right)a_{21} + \left(\tfrac{5}{24} + \tfrac{5\sqrt{5}}{120}\right)\left(a_{31} + a_{32}\right) + \tfrac{1}{12}\left(a_{41} + a_{42} + a_{43}\right) = \tfrac{1}{144}. \quad (48)$$

Solving (43)-(48) simultaneously we get

$$a_{21} = \tfrac{1}{168} + \tfrac{11\sqrt{5}}{4200}, a_{31} = \tfrac{1}{420} - \tfrac{\sqrt{5}}{700}, a_{32} = \tfrac{1}{280} - \tfrac{\sqrt{5}}{840},$$

$$a_{41} = -\tfrac{1}{840} - \tfrac{\sqrt{5}}{168}, a_{42} = \tfrac{1}{84} - \tfrac{\sqrt{5}}{140}, a_{43} = \tfrac{5}{168} + \tfrac{11\sqrt{5}}{840}.$$

Consequently we compute the local truncation errors and is given as follows:

$$\left\|\tau^{(7)}\right\|_2 = 0, \left\|\tau'^{(7)}\right\|_2 = 2.380952381 \times 10^{-4},$$

$$\left\|\tau''^{(7)}\right\|_2 = 4.761904762 \times 10^{-4}, \left\|\tau'''^{(7)}\right\|_2 = 4.761904762 \times 10^{-4},$$

and the global error norm is $\left\|\tau_g^{(7)}\right\|_2 = 7.142857143 \times 10^{-4}$.

where $\tau^{(7)}, \tau'^{(7)}, \tau''^{(7)}$ and $\tau'''^{(7)}$ are error norms for y, y', y'' and y''' of the seventh order conditions of the RKFD6

TABLE 1. Butcher tableau for RKFD6 method

0	0			
$\frac{1}{2} + \frac{\sqrt{5}}{10}$	$\frac{1}{168} + \frac{11\sqrt{5}}{4200}$			
$\frac{1}{2} - \frac{\sqrt{5}}{10}$	$\frac{1}{420} - \frac{\sqrt{5}}{700}$	$\frac{1}{280} - \frac{\sqrt{5}}{840}$		
1	$-\frac{1}{840} - \frac{\sqrt{5}}{168}$	$\frac{1}{84} - \frac{\sqrt{5}}{140}$	$\frac{5}{168} + \frac{11\sqrt{5}}{840}$	
	$\frac{17}{1260}$	$\frac{1}{72} - \frac{\sqrt{5}}{168}$	$\frac{1}{72} + \frac{\sqrt{5}}{168}$	$\frac{1}{2520}$
	$\frac{1}{24}$	$\frac{1}{16} - \frac{\sqrt{5}}{48}$	$\frac{1}{16} + \frac{\sqrt{5}}{48}$	0
	$\frac{1}{12}$	$\frac{5}{24} - \frac{\sqrt{5}}{24}$	$\frac{5}{24} + \frac{\sqrt{5}}{24}$	0
	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

method, respectively. Finally, all the parameters of four-stage sixth-order RKFD method and denoted as RKFD6 are written in Butcher tableau as follows (Table 1):

NUMERICAL RESULTS

In order to evaluate the performance of the new RKFD6 method, a set of special fourth-order ODEs are solved and then compared the results with some efficient methods, which are chosen from the scientific literature. We use in the numerical comparisons the criteria based on the maximum error in the solution (*Max Error* = max($|y(x_n)$ $- y_n|$)) which is equal to the maximum of absolute errors of the true solutions and the computed solutions. Figures 1-5 show the efficiency curves of Log_{10}(*Max Error*) against the computational effort measured by Log_{10}(*Function Evaluations*) which are required by each method. The following methods are used in comparison: RKFD6: The four-stage sixth-order RKFD method derived in this paper; RK6N: The seven-stage sixth-order RK method given in Butcher (1964); and RK6B: The seven-stage sixth-order RK method derived by Butcher (2008).
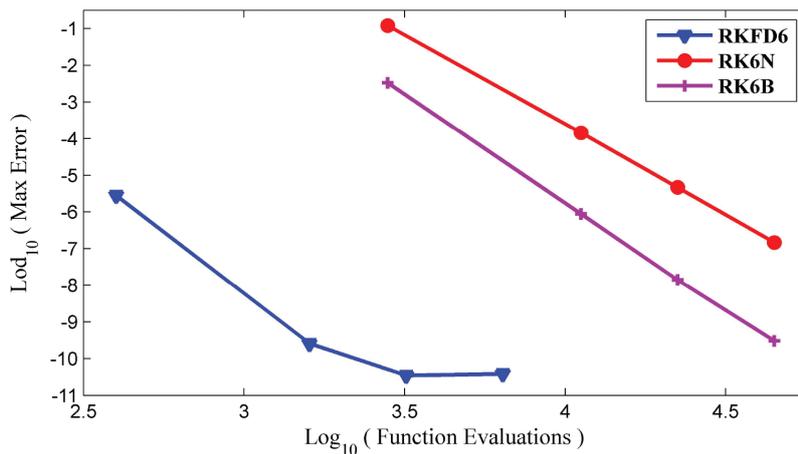
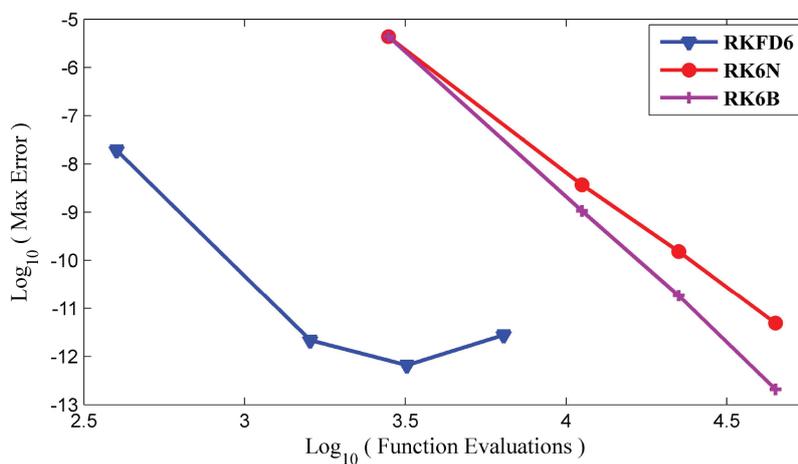FIGURE 1. The efficiency curves for Problem 1 with $h = 0.1/(2^i, i = 0, 2, 3, 4)$

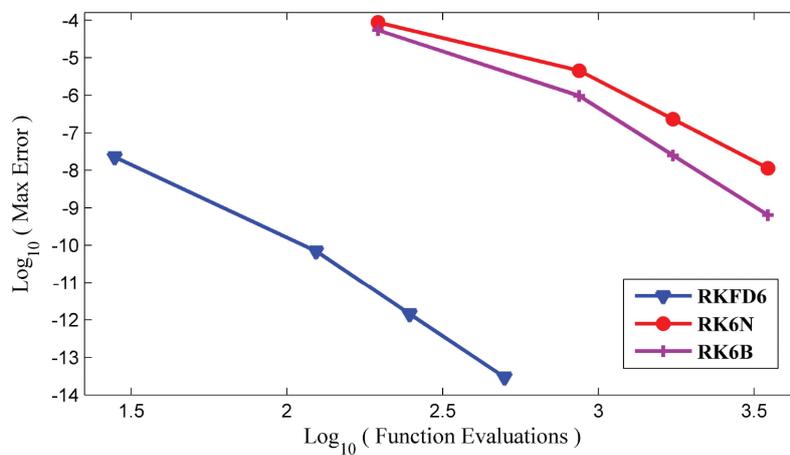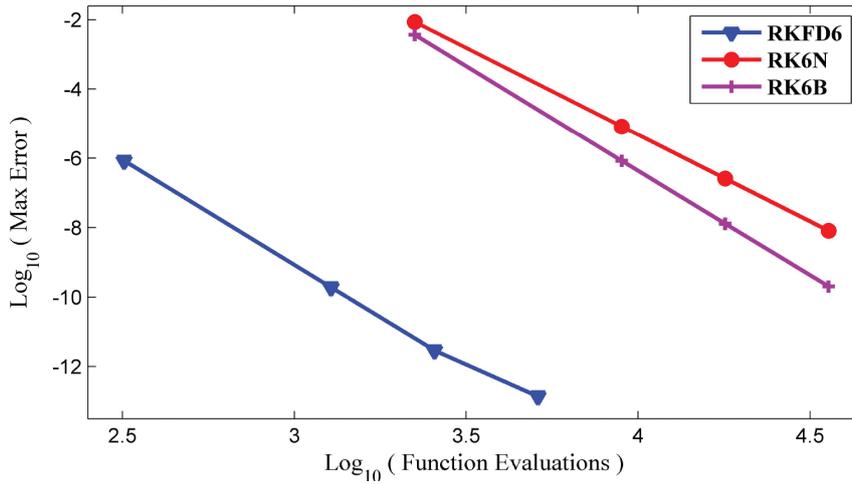FIGURE 2. The efficiency curves for Problem 2 with $h = 0.1/(2^i, i = 0, 2, 3, 4)$

FIGURE 3. The efficiency curves for Problem 3 with $h = 0.1/(2^i, \; i = 0, 2, 3, 4)$

FIGURE 4. The efficiency curves for Problem 4 with $h = 0.1/(2^i, i = 0, 2, 3, 4)$
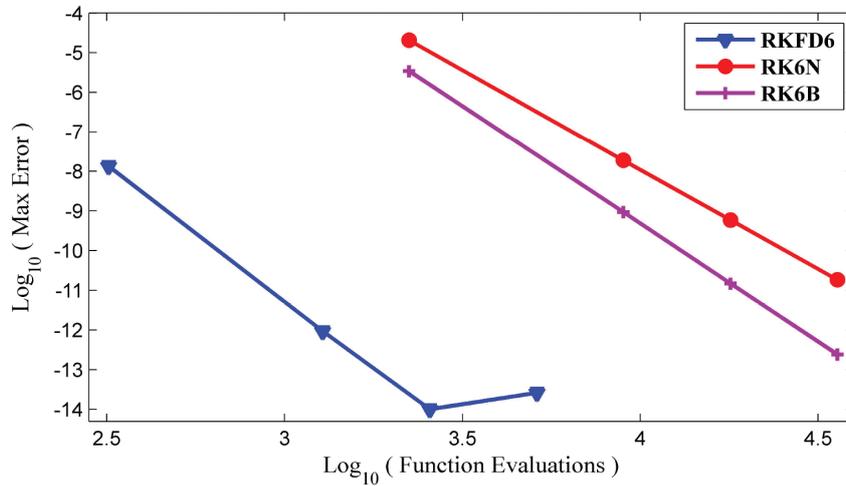

FIGURE 5. The efficiency curves for Problem 5 with $h = 0.1/(2^i, i = 0, 2, 3, 4)$

Problem 1 Consider the homogeneous linear problem:

$$y^{(iv)} = -4y, \quad 0 < x \le 10,$$

$$y(0) = 0, \ y'(0) = 1, \ y''(0) = 2, \ y'''(0) = 2.$$

The exact solution is given by $y(x) = e^x \sin(x)$.

Problem 2 Consider the non-homogeneous non-linear problem:

$$y^{(iv)} = y^2 + cos^2(x) + \sin(x) - 1, \quad 0 < x \le 10,$$

$$y(0) = 0, \ y'(0) = 1, \ y''(0) = 0, y'''(0) = -1.$$

The exact solution is given by $y(x) = \sin(x)$.

Problem 3 Consider the non-linear problem:

$$y^{(iv)} = \frac{3\sin(y)\left(3 + 2\sin^2(y)\right)}{cos^7(y)},$$

$$y(0) = 0, \ y'(0) = 1, \ y''(0) = 0, \ y'''(0) = 1.$$

The exact solution is given by $y(x) = arc\sin(x)$. The problem is integrated in the interval $\left[0, \frac{\pi}{4}\right]$.

Problem 4 Consider the linear system:

$$y^{(iv)} = e^{3x} u, \ y(0) = 1, y'(0) = -1, \ y''(0) = 1, \ y'''(0) = -1.$$

$$z^{(iv)} = 16e^{-x} y, \ z(0) = 1, z'(0) = -2, \ z''(0) = 4, \ z'''(0) = -8.$$

$w^{(iv)} = 81e^{-x} z, w(0) = 1, w'(0) = -3, w''(0) = 9,$
$w'''(0) = -27.$

$u^{(iv)} = 256e^{-x} w, \quad u(0) = 1, u'(0) = -4, \quad u''(0) = 16,$
$u'''(0) = -64.$

The problem is integrated in the interval [0, 2]. The exact solution is given by:

$$y = e^{-x}, \; z = e^{-2x}, \; w = e^{-3x}, \; u = e^{-4x}.$$

Problem 5 Consider the non-linear system:

$$y^{(iv)} = y + \frac{1}{\sqrt{y^2 + z^2}} - \frac{1}{\sqrt{w^2 + u^2}},$$

$y(0) = 1, y'(0) = 0, \; y''(0) = -1, \; y'''(0) = 0.$

$$z^{(iv)} = z - \frac{1}{\sqrt{y^2 + z^2}} + \frac{1}{\sqrt{w^2 + u^2}},$$

$z(0) = 0, z'(0) = 1, \; z''(0) = 0, \; z'''(0) = -1.$

$$w^{(iv)} = 16w + \frac{1}{\sqrt{y^2 + z^2}} - \frac{1}{\sqrt{w^2 + u^2}},$$

$w(0) = 1, w'(0) = 0, \; w''(0) = -4, \; w'''(0) = 0.$

$$u^{(iv)} = 16u - \frac{1}{\sqrt{y^2 + z^2}} + \frac{1}{\sqrt{w^2 + u^2}},$$

$u(0) = 0, u'(0) = 2, \; u''(0) = 0, \; u'''(0) = -8.$

The problem is integrated in the interval [0, 2]. The exact solution is given by:

$$y = \cos(x), \; z = \sin(x), \; w = \cos(2x), \; u = \sin(2x).$$

CONCLUSION

In this paper, we have derived the order conditions of the RKFD methods up to order seven. Based on the order conditions, we constructed the four-stage sixth-order RKFD method for directly solving special fourth-order ODEs, this method denoted as RKFD6 method. We also proved that the RKFD method is zero-stable. Numerical results illustrate that the RKFD6 method is more efficient as compared with the well known Runge-Kutta methods in the literature. As a conclusion it can be said that, the new RKFD6 method can efficiently solve the special fourth-order ODEs in terms of accuracy and number of function evaluations.

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