

## Homotopy Decomposition Method for Solving Higher-Order Time-Fractional Diffusion Equation via Modified Beta Derivative

(Kaedah Penguraian Homotopi bagi Menyelesaikan Persamaan Resapan Pecahan-Masa Peringkat Tinggi menerusi Terbitan Terubah Suai Beta)

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### ABSTRACT

*In this paper, the homotopy decomposition method with a modified definition of beta fractional derivative is adopted to find approximate solutions of higher-dimensional time-fractional diffusion equations. To apply this method, we find the modified beta integral for both sides of a fractional differential equation first, then using homotopy decomposition method we can obtain the solution of the integral equation in a series form. We compare the solutions obtained by the proposed method with the exact solutions obtained using fractional variational homotopy perturbation iteration method via modified Riemann-Liouville derivative. The comparison shows that the results are in a good agreement.*

*Keywords: Beta derivative; fractional differential equation; fractional diffusion equation; homotopy decomposition method*

### ABSTRAK

*Dalam kertas ini, kaedah penguraian homotopi dengan takrif terbitan pecahan beta terubah suai diadaptasi untuk mencari penyelesaian penghampiran bagi persamaan resapan pecahan-masa peringkat tinggi. Untuk menggunakan kaedah ini, kami dapatkan dahulu kamiran beta terubah suai bagi kedua-dua belah persamaan terbitan pecahan itu, kemudian dengan menggunakan kaedah penguraian homotopi kami boleh dapatkan penyelesaian bagi persamaan kamiran itu dalam bentuk siri. Kami bandingkan penyelesaian yang diperolehi dengan penyelesaian tepat yang diperolehi menerusi kaedah usikan berlelar homotopi ubahan dengan terbitan Riemann-Liouville. Perbandingan menunjukkan kedua-dua kaedah memberikan penyelesaian yang sangat hampir.*

*Kata kunci: Kaedah penguraian homotopi; persamaan resapan pecahan; persamaan terbitan pecahan; terbitan beta*

### INTRODUCTION

Fractional calculus has become very useful in applied mathematics. Many natural phenomena have to be described by means of fractional (non-integer) derivatives instead of the classical (integer) derivatives: the first reason for this choice is that modeling physical and engineering processes via fractional derivatives give us numerous and universal cases of the model rather than one special case when using integer derivatives. For example, Abdullah (2013) used fractional differential equations to perform a simulation model of Hirschsprung's disease (HSCR). Fractional derivatives give the history effects of a phenomenon, which is usually neglected completely in the integer derivatives. Different types of fractional derivatives and their properties were studied (Kilbas et al. 2006; Miller & Ross 1993; Oldham & Spanier 1974; Podlubny 1999; Samko et al. 1993).

Fractional partial differential equations (FPDEs) can be divided into two main parts: Space-fractional differential equations and time-fractional differential equations. The time-fractional diffusion equation (TFDE) can be obtained from the standard diffusion equation by replacing the first-order time derivative with a

fractional derivative of order  $\alpha$ , with  $0 \leq \alpha < 1$  (Lin & Xu 2007). The fractional diffusion equation (TFDE) and the fractional advection-dispersion equation (TFADE) have been commonly researched by Huang and Liu (2005). From a physical point of view, TFADE originates from a fractional Fick law switching the classical Fick law, which describes transport processes with a long memory (Gorenflo et al. 2002). Nigmatullin (1986, 1984) indicated that many of the general electromagnetic, acoustic and mechanical responses can be modeled precisely using fractional diffusion-wave equations. For instance, a TFDE has been explicitly presented in physics by Nigmatullin (1986) to define diffusion in special types of porous media which exhibit a fractal geometry. The solution of TFDE cannot always be found analytically; so it is necessary to use numerical methods. Li et al. (2009) solved TFDE with a moving boundary condition using homotopy perturbation method. Çetinkaya and Kıymaz (2013) used the generalized differential transform method to obtain the solution of TFDE. Ray and Bera (2006) have used the Adomian decomposition method (ADM) to find the solution of a TFDE of order  $\beta = 1/2$ . The variational iteration method (VIM) has been used for the same

equation by Das (2009a) with the initial conditions 1,  $x$  and  $x^2$ . TFDE of order  $\beta$ , ( $0 < \beta \leq 1$ ) with the initial conditions  $x^n$ ,  $n \in \mathbb{N}$  has been solved by Das (2009b).

Finding an appropriate analytical and numerical technique for solving fractional differential equations is considered as one of the big challenges for researchers. To deal with this problem, many scholars try to adjust the classical methods for solving ordinary and partial differential equations to be suitable for solving fractional differential equations. The homotopy decomposition method (HDM) has been proposed by Atangana and Botha (2012) to solve a partial differential equation that arises in groundwater flow problems. This method was first used to solve time-fractional coupled-Korteweg-de Vries equations by Atangana and Secer (2013). Atangana and Belhaouari (2013) proposed analytical solution of the partial differential equation with time and space-fractional derivatives using this method.

The main aim of this work is to develop a modified method for the beta derivative based on the HDM to solve TFDE. This study is arranged as follows: After giving basic descriptions of significant functions and fractional derivatives in the next section, analysis of homotopy decomposition method via beta derivative is discussed next. Then subsequently, to demonstrate the accuracy of this method, we find approximate solutions for fractional diffusion equation. Conclusions are set in the final section.

#### PRELIMINARIES AND FRACTIONAL DERIVATIVE ORDER

This section gives important definitions that will be used in the following sections. Firstly, we will indicate two most important functions that appear frequently in the fractional derivatives: the Mittag-Leffler function and the Gamma function, then the basic definitions of the fractional derivatives will be presented.

##### MITTAG-LEFFLER FUNCTION

The Mittag-Leffler (ML) function is a direct generality of the exponential function,  $e^x$ , and it represents a major part in fractional calculus. The one-parameter function in powers series is given by the formula (Podlubny 1999),

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad (\alpha > 0) \quad (1)$$

##### THE GAMMA FUNCTION

The most basic interpretation of the (complete) Gamma function  $\Gamma(x)$  is simply the generalization of the factorial to complex and real arguments. The Gamma function can be defined as (Podlubny 1999),

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \quad x > 0. \quad (2)$$

##### FRACTIONAL DERIVATIVE ORDER

There are several definitions of the fractional derivatives. The two most commonly used definitions for the general

fractional derivative are: the Riemann-Liouville derivative and Caputo fractional derivative.

The Riemann-Liouville (RL) derivative is defined as (Samko et al. 1993),

$$D_a^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n, \\ \frac{d^n}{dt^n} f(t), & \alpha = n, \end{cases} \quad (3)$$

where  $\alpha > 0$ ,  $t > a$ ,  $n \in \mathbb{N}$  and  $\alpha, a, t \in \mathbb{R}$ .

The Caputo fractional derivative is defined as (Samko et al. 1993),

$$D_a^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n, \\ \frac{d^n}{dt^n} f(t), & \alpha = n, \end{cases} \quad (4)$$

where  $\alpha > 0$ ,  $t > a$ ,  $n \in \mathbb{N}$  and  $\alpha, a, t \in \mathbb{R}$ .

Jumarie (2006) proposed a modified Riemann-Liouville derivative (mRL), called the Jumarie derivative, defined as

$$D^{\alpha} f(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} [f(\tau) - f(0)] d\tau, \quad 0 < \alpha < 1. \quad (5)$$

Note that these three definitions of fractional derivatives have their benefits and drawbacks. For the RL fractional derivative, an arbitrary function needs not to be continuous at the origin and it will not be differentiable. One weakness of the RL when trying to model real-life phenomena is that the RL derivative of a constant is not zero. Besides, if an arbitrary function is a constant at the origin, its fractional derivative has a singularity at the origin, for example the exponential and ML functions. On the other hand, one of the benefits of the Caputo fractional derivative is that it agrees with the usual initial and boundary conditions to be involved in the formulation of the problem. Also, its derivative for a constant is zero. Caputo's derivative has the downside that to estimate the fractional derivative of a function in the Caputo sense, we are required to determine its derivative first. Caputo derivatives are well-defined for differentiable functions, while functions with no first-order derivative might have fractional derivatives of all orders less than one in the RL sense. In the mRL fractional derivative, an arbitrary continuous function will not be differentiable; the fractional derivative of a constant is equal to zero and more significantly it removes singularity at the origin for all functions, for instance, the exponential functions and ML functions. Also, with the Jumarie derivative, there is a famous drawback, especially for not continuous functions at the origin, so the fractional derivative does not exist (Atangana 2015). Atangana (2015) proposed a new definition of a fractional derivative called  $\beta$ -derivative for trying to overcome some of the previous disadvantages and it uses limit instead of integral to define the fractional order derivative. The  $\beta$ -derivative is defined as (Atangana 2015),

Let  $a \in \mathbb{R}$  and  $m$  be a function, such that,  $m : [a, \infty) \rightarrow \mathbb{R}$ . We can define  $\beta$ -derivative as

$$D_t^\beta m(t) := \begin{cases} \lim_{\varepsilon \rightarrow 0} \frac{m\left(t + \varepsilon\left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - m(t)}{\varepsilon}, & t \geq 0, 0 < \beta \leq 1, \\ m(t), & t \geq 0, \beta = 0. \end{cases} \quad (6)$$

If the above limit exists, then  $m$  is said to be  $\beta$ -differentiable. For  $\beta = 1$ , we have  $D_t^\beta m(t) = dm(t)/dt$ .

Unfortunately, the  $\beta$ -derivative provides us in several cases divergent solutions or vary extreme approximate solutions in comparison with the exact solutions. Therefore, to solve this problem in  $\beta$ -derivative we improve the definition to deal with different kinds of fractional differential equations with HDM. The results show that this suggested definition is a promising one.

#### ANALYSIS OF THE HDM

The HDM is the connection of the Cauchy formula of  $n$ -integral together with the idea of homotopy. To describe the idea of this method, consider a general fractional partial differential equation (Atangana 2015),

$${}^A D_t^\beta (\Psi(x, t)) = \mathbf{L}(\Psi(x, t)) + \mathbf{N}(\Psi(x, t)) + h(x, t), 0 < x \leq 1, \quad (7)$$

with initial condition

$$\Psi(x, 0) = g(x), \quad (8)$$

where  ${}^A D_t^\beta$  indicates the  $\beta$ -derivative operator;  $g$  is a known function;  $\mathbf{N}$  is the nonlinear fractional differential operator; and  $\mathbf{L}$  denotes a linear fractional differential operator. By  $\beta$ -derivative, we apply the inverse operator  ${}^A I_t^\beta$  to transform the fractional differential (7) to the fractional integral equation,

$$\Psi(x, t) - \Psi(x, 0) = \int_0^x \left(y + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} (\mathbf{L}(\Psi(x, t)) + \mathbf{N}(\Psi(x, t)) + h(x, t)) dy. \quad (9)$$

The solution in the homotopy perturbation method can be expressed as,

$$\Psi(x, t, \mathbf{p}) = \sum_{n=0}^{\infty} \mathbf{p}^n \Psi_n(x, t), \quad \Psi(x, t) = \lim_{\mathbf{p} \rightarrow 1} \Psi(x, t, \mathbf{p}). \quad (10)$$

While, the nonlinear term can be decomposed as,

$$\mathbf{N}(\Psi(x, t)) = \sum_{n=0}^{\infty} \mathbf{p}^n H_n(\Psi), \quad (11)$$

where  $\mathbf{p} \in (0, 1]$  is an embedding parameter.  $H_n(\Psi)$  is He's polynomials (He 1999) and can be generated by,

$$H_n(\Psi_0, \Psi_1, \dots, \Psi_n) = \frac{1}{n!} \frac{\partial^n}{\partial \mathbf{p}^n} \left[ \mathbf{N} \left( \sum_{j=0}^{\infty} \mathbf{p}^j \Psi_j(x, t) \right) \right],$$

$$n = 0, 1, 2, \dots \quad (12)$$

Substituting (11), (12) and (10) into (9), we have

$$\sum_{n=0}^{\infty} \mathbf{p}^n \Psi_n(x, t) - \Psi(x, 0) = \mathbf{p} \int_0^x \left(y + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} (\mathbf{L}(\sum_{n=0}^{\infty} \mathbf{p}^n \Psi_n(x, t)) + \mathbf{N}(\sum_{n=0}^{\infty} \mathbf{p}^n \Psi_n(x, t)) + h(x, t)) dy. \quad (13)$$

Comparison of the terms of the same powers of  $\mathbf{p}$  gives solutions of various orders with the first term,

$$\Psi_0(x, t) = \Psi(x, 0). \quad (14)$$

In this work, we present the modified  $\beta$  ( $m\beta$ )-derivative, then (13) becomes,

$$\sum_{n=0}^{\infty} \mathbf{p}^n \Psi_n(x, t) - \Psi(x, 0) = \frac{\mathbf{p}}{\Gamma(\beta+n)} \int_0^x \left(y + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} [\mathbf{L}(\sum_{n=0}^{\infty} \mathbf{p}^n \Psi_n(x, t)) + \mathbf{N}(\sum_{n=0}^{\infty} \mathbf{p}^n \Psi_n(x, t)) + h(x, t)] dy. \quad (15)$$

#### TEST EXAMPLES

To demonstrate the advantage of the modified definition we propose HDM to find approximate solution of TFDE of the form,

$$\frac{\partial^\beta u(X, t)}{\partial t^\beta} = D \Delta u(X, t) - \nabla \cdot F(X) u(X, t), \quad 0 < \beta \leq 1, D > 0, \quad (16)$$

subject to initial and boundary conditions

$$u(X, 0) = \varnothing(X), X \in \Omega, \quad (17)$$

$$u(X, t) = \varphi(X, t), X \in \partial\Omega, t \geq 0. \quad (18)$$

Here,  $\partial\beta/\partial t\beta(\cdot)$  is the  $m\beta$ -derivative of order  $\beta$ .  $\Delta$  is the Laplace operator,  $\nabla$  is the Hamilton operator,  $\Omega = [0, L_1] \times [0, L_2] \times \dots \times [0, L_d]$  is the spatial domain of the problem,  $d$  is the dimension of the space,  $X = (x_1, x_2, \dots, x_d)$ ,  $\partial\Omega$  is the boundary of  $\Omega$ ,  $u(X, t)$  denotes the probability density function of finding a particle at  $X$  in time  $t$ , the positive constant  $D$  depends on the temperature, the friction coefficient, the universal gas constant and lastly on the Avagadro constant,  $F(X)$  is the external force. Equation (16) can be interpreted as a model of the diffusion of a particle under the action of the external force  $F(X)$  (Guo et al. 2013).

EXAMPLE 1. TWO-DIMENSIONAL CASE

Let  $D = 1$ ,  $\Omega = [0,1] \times [0,1]$ ,  $F = -(x, y)$  in (16), then we have the following TFDE:

$$\frac{\partial^\beta u(x, y, t)}{\partial t^\beta} = \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} + x \frac{\partial u(x, y, t)}{\partial x} + y \frac{\partial u(x, y, t)}{\partial y} + 2u(x, y, t), 0 < \beta \leq 1, \tag{19}$$

with initial condition

$$u(x, y, 0) = x + y. \tag{20}$$

Equation (19) has been solved using fractional variational homotopy perturbation iteration method (FVHPIM) via  $m - R - L$  derivative (Guo et al. 2013) and the exact solution is

$$u(x, y, t) = (x + y) E_\beta(3t^\beta), \tag{21}$$

where  $E_\beta(x)$  is the ML function defined in (1).

To solve (19) using the HDM via  $m\beta$ -derivative, we apply on both sides the  $m - \beta$  -integral to obtain,

$$\sum_{n=0}^{\infty} \mathbf{P}^n u_n(x, y, t) - u(x, y, 0) = \frac{\mathbf{P}}{\Gamma(\beta+n)} \times \int_0^t \left( \tau + \frac{1}{\Gamma(\beta)} \right)^{\beta-1} \left[ \left( \sum_{n=0}^{\infty} \mathbf{P}^n u_n(x, y, \tau) \right)_{x,x} + \left( \sum_{n=0}^{\infty} \mathbf{P}^n u_n(x, y, \tau) \right)_{y,y} + x \left( \sum_{n=0}^{\infty} \mathbf{P}^n u_n(x, y, \tau) \right)_x + y \left( \sum_{n=0}^{\infty} \mathbf{P}^n u_n(x, y, \tau) \right)_y + 2 \sum_{n=0}^{\infty} \mathbf{P}^n u_n(x, y, \tau) \right] d\tau. \tag{22}$$

Comparing the terms of  $\mathbf{p}$  the same power of and using (16), we have the following integral equations,

$$\mathbf{p}^0 : u_0(x, y, t) = u(x, y, 0),$$

$$\mathbf{p}^1 : u_1 = \frac{1}{\Gamma(\beta+1)} \int_0^t \left( \frac{1}{\Gamma(\beta)+\tau} \right)^{\beta-1} \left( \frac{d^2 u_0}{dx^2} + \frac{d^2 u_0}{dy^2} + x \frac{du_0}{dy} + y \frac{du_0}{dx} + 2u_0 \right) d\tau, u_1(x, y, 0) = 0$$

$$\mathbf{p}^2 : u_2 = \frac{1}{\Gamma(\beta+2)} \int_0^t \left( \frac{1}{\Gamma(\beta)+\tau} \right)^{\beta-1} \left( \frac{d^2 u_1}{dx^2} + \frac{d^2 u_1}{dy^2} + x \frac{du_1}{dx} + y \frac{du_1}{dy} + 2u_1 \right) d\tau, u_2(x, y, 0) = 0$$

⋮

$$\mathbf{P}^n : u_n = \frac{1}{\Gamma(\beta+n)} \int_0^t \left( \frac{1}{\Gamma(\beta)+\tau} \right)^{\beta-1} \left( \frac{d^2 u_{n-1}}{dx^2} + \frac{d^2 u_{n-1}}{dy^2} + x \frac{du_{n-1}}{dx} + y \frac{du_{n-1}}{dy} + 2u_{n-1} \right) d\tau, u_n(x, y, 0) = 0.$$

Then, we find the following series solution

$$u_0 = x + y, u_1 = \frac{3\Gamma(\beta)^{-\beta-1}(x+y)(t\Gamma(\beta)+1)^\beta - 1}{\beta^2}, u_2 = \frac{9\Gamma(\beta)^{-2\beta-1}(x+y)(t\Gamma(\beta)+1)^\beta - 1^2}{\beta^3 \Gamma(\beta+2)}, u_3 = \frac{27\Gamma(\beta)^{-3\beta-1}(x+y)(t\Gamma(\beta)+1)^\beta - 1^3}{\beta^4 \Gamma(\beta+2)\Gamma(\beta+3)}, \dots u_k = \frac{3^k \Gamma(\beta)^{-k\beta-1}(x+y)(t\Gamma(\beta)+1)^\beta - 1^k}{\beta^{k+1} \prod_{m=2}^k \Gamma(\beta+m)},$$

for  $k \geq 2$ .

When  $\beta = 1$  the sum of the first five terms of approximate solution will be

$$\sum_{n=0}^4 u_n(x, y, t) = x + y + 3t(x + y) + \frac{9}{2} t^2 (x + y) + \frac{9}{4} t^3 (x + y) + \frac{9}{32} t^4 (x + y). \tag{23}$$

We can compare the tenth-order approximate solution with the exact solution (21). Figure 1 gives the comparison of tenth-order approximate solution using HDM via  $m\beta$ -derivative, with the exact solution ( $\beta = 1$ ) using FVHPIM via mRL derivative. Meanwhile, Figure 2 shows the tenth-order approximate solutions for different values of  $\beta$  via  $m\beta$ -derivative.

EXAMPLE 2. THREE-DIMENSIONAL CASE

Let  $D = 1$ ,  $\Omega = [0,1] \times [0,1] \times [0,1]$ ,  $F(x, y, z) = -(x, y, z)$  in (16), then we have the following TFDE:

$$\frac{\partial^\beta u(x, y, z, t)}{\partial t^\beta} = \Delta u(x, y, z, t) + x \frac{\partial u(x, y, z, t)}{\partial x} + y \frac{\partial u(x, y, z, t)}{\partial y} + z \frac{\partial u(x, y, z, t)}{\partial z} + 3u(x, y, z, t), 0 < \beta \leq 1, \tag{24}$$

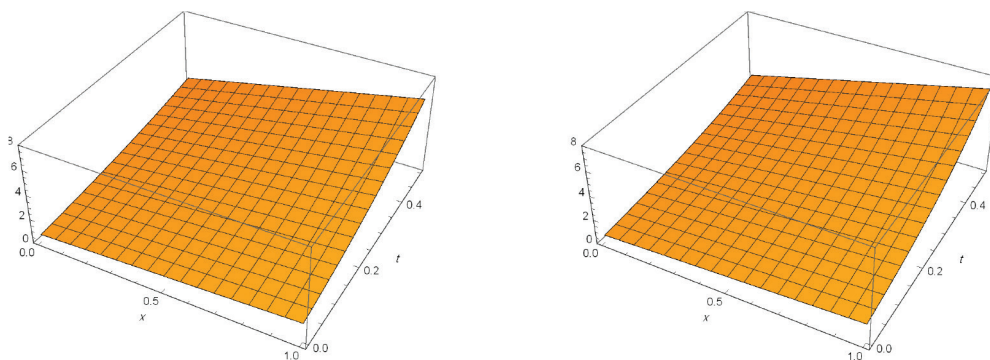


FIGURE 1. (left) The tenth-order approximate solution. (right) The exact solution ( $\beta = 1$ ), ( $y = 0.6$ )

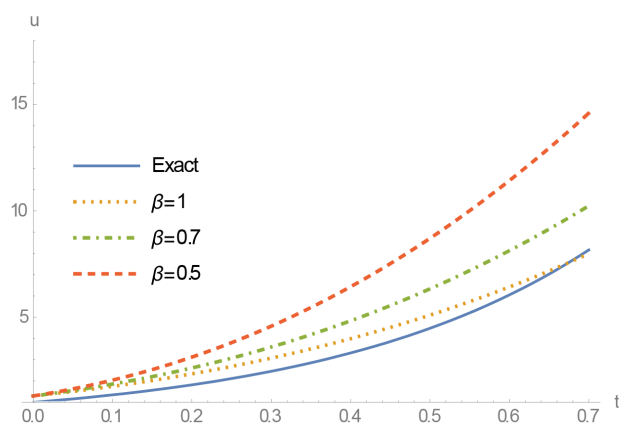


FIGURE 2. The exact solution ( $\beta = 1$ ) and the tenth-order approximate solution for different values of  $\beta$ , ( $x = 0.4$ ,  $y = 0.9$ )

with the initial condition,

$$u(x, y, z, 0) = (x + y + z)^2. \tag{25}$$

Equation (24) has been solved using FVHPIM via mRL derivative (Guo et al. 2013) and the exact solution is,

$$u(x, y, z, t) = ((3 + (x+y+z)2)) E_\beta(5t^\beta) - 3E_\beta(3t^\beta), \tag{26}$$

where  $E_\beta(x)$  is the ML function defined in (1).

Following the same procedures as in the previous example, we can find the following series solutions:

$$u_0 = (x + y + z)^2,$$

$$u_1 = \frac{\Gamma(\beta)^{-\beta} ((t\Gamma(\beta)+1)^\beta - 1)(5(x+y+z)^2 + 6)}{\beta\Gamma(\beta+1)},$$

$$u_2 = \frac{\Gamma(\beta)^{1-2\beta} ((t\Gamma(\beta)+1)^\beta - 1)^2 (25(x+y+z)^2 + 48)}{\beta\Gamma(\beta+1)^2 \Gamma(\beta+2)},$$

$$u_3 = \frac{\Gamma(\beta)^{2-3\beta} ((t\Gamma(\beta)+1)^\beta - 1)^3 (125(x+y+z)^2 + 294)}{\beta\Gamma(\beta+1)^3 \Gamma(\beta+2) \Gamma(\beta+3)},$$

⋮

$$u_k = \frac{\Gamma(\beta)^{(k-1)-k\beta} ((t\Gamma(\beta)+1)^\beta - 1)^k (5^k(x+y+z)^2 + 3(5^k - 3^k))}{\beta\Gamma(\beta+1)^k \prod_{m=2}^k \Gamma(\beta+m)},$$

for  $k \geq 2$ .

When  $\beta = 1$  the sum of the first five terms of approximate solution will be:

$$\begin{aligned} & \sum_{n=0}^4 u_n(x, y, z, t) \\ &= \frac{1}{288} t^4 (625(x+y+z)^2 + 1632) \\ &+ \frac{1}{12} t^3 (125(x+y+z)^2 + 294) + \frac{1}{2} t^2 (25(x+y+z)^2 + 48) \\ &+ t(5(x+y+z)^2 + 6) + (x+y+z)^2. \end{aligned}$$

Similar to Example 1, we can compare the tenth-order approximate solution using HDM via  $m\beta$ -derivative with the exact solution using FVHPIM via mRL derivative when  $\beta = 1$  in Figure 3. Meanwhile, Figure 4 shows the tenth-order approximate solutions for different values of  $\beta$ .

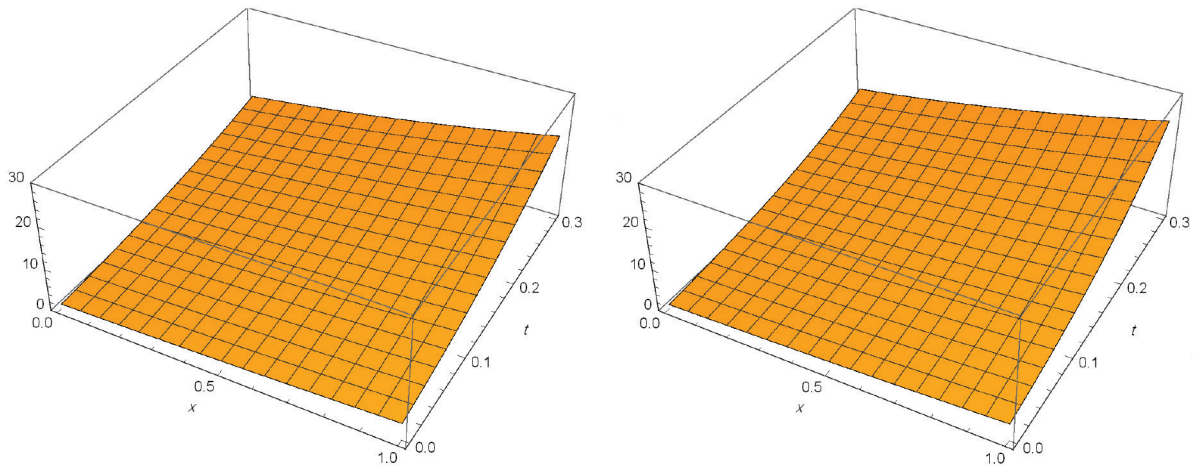


FIGURE 3. (left) The tenth-order approximate solution. (right) The exact solution ( $\beta = 1$ ), ( $x = 0.5$ ,  $z = 0.5$ )

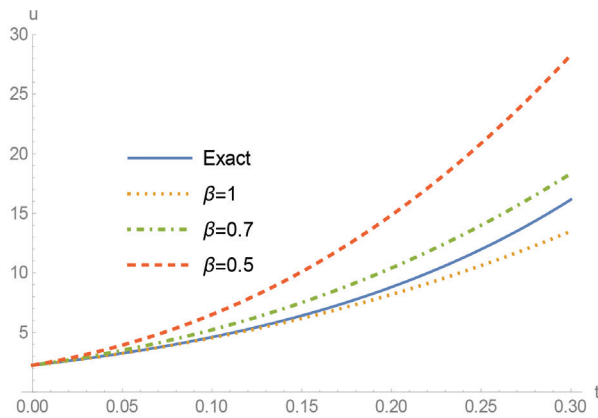


FIGURE 4. The exact solution ( $\beta = 1$ ) and the tenth-order approximate solution for different values of  $\beta$ , ( $x = 0.5$ ,  $y = 0.5$ ,  $z = 0.7$ )

#### CONCLUSION AND DISCUSSION

The homotopy decomposition method (HDM) via modified beta derivative has been fruitfully applied to find the approximate solution to higher dimensions of time-fractional diffusion equations. Figures 1 and 3 show that the graphical representations of tenth-order approximate solutions of the time-fractional diffusion equations using proposed method are very close to the exact solutions obtained by FVHPIM via modified Riemann-Liouville derivative when  $\beta = 1$ . In Figures 2 and 4 we compare the exact solutions ( $\beta = 1$ ) with different values of  $\beta$  in tenth-order approximate solutions. The results indicated that the suggested method is a promising mathematical tool for finding the approximate numerical solutions for different types of fractional differential equations.

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