

Block Hybrid Method with Trigonometric-Fitting for Solving Oscillatory Problems (Kaedah Blok Hibrid dengan Penyuaian-Trigonometri untuk Menyelesaikan Masalah Berayun)

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ABSTRACT

In this paper, we develop algebraic order conditions for two-point block hybrid method up to order five using the approach of B-series. Based on the order conditions, we derive fifth order two-point block explicit hybrid method for solving special second order ordinary differential equations (ODEs), where the existing explicit hybrid method of order five is used to be the method at the first point. The method is then trigonometrically fitted so that it can be suitable for solving highly oscillatory problems arising from special second order ODEs. The new trigonometrically-fitted block method is tested using a set of oscillatory problems over a very large interval. Numerical results clearly showed the superiority of the method in terms of accuracy and execution time compared to other existing methods in the scientific literature.

Keywords: B-Series; explicit block hybrid method; oscillatory problems

ABSTRAK

Dalam kertas ini, kami membangunkan syarat peringkat aljabar kaedah blok hibrid dua titik sehingga peringkat kelima menggunakan pendekatan siri-B. Berdasarkan syarat peringkat tersebut, kami menerbitkan kaedah blok hibrid tak tersirat dua titik peringkat kelima untuk menyelesaikan persamaan pembezaan biasa (PPB) khas peringkat kedua, dengan kaedah hibrid tak tersirat sedia ada peringkat kelima digunakan sebagai kaedah pada titik pertama. Kaedah ini kemudiannya difasa-suaikan secara trigonometri supaya sesuai untuk menyelesaikan masalah berayun yang timbul daripada persamaan pembezaan khas peringkat kedua. Kaedah baru blok trigonometri fasa-suai ini diuji menggunakan satu set masalah berayun bagi selang yang sangat besar. Keputusan berangka dengan jelas menunjukkan keunggulan kaedah tersebut daripada segi ketepatan dan masa pengiraan berbanding kaedah sedia ada yang lain dalam kepustakaan saintifik.

Kata kunci: Kaedah blok hibrid tak tersirat; masalah berayun; Siri-B

INTRODUCTION

In this paper, we focus on special second order ordinary differential equations (ODEs) in the form,

$$y'' = f(x, y), y(x_0) = x_0, y'(x_0) = y_0', \quad (1)$$

in which the function does not depend on the derivative of y . A class of this type of differential equations is known to be highly oscillatory, further details can be obtained in Papadopoulos et al. (2009), hence it is quite difficult to get their numerical solutions accurately. When solving oscillatory problems, there are two types of errors that we need to focus on apart from the local truncation error that is due to the algebraic order of the method. They are phase-lag error which is defined as the difference of the angles of the computed solution and the exact solution and dissipation error, which is the distance of the computed solution from the cyclic solution. Numerical methods can adapt to the special structure of the oscillatory problem, such techniques, for that purpose, are trigonometrically-fitted and phase-fitted techniques. Franco (1995), in his work, has proposed that (1) can directly be solved using explicit hybrid algorithms or special multistep methods for second-order ODEs. By using the local truncation

error and the algebraic order conditions from Coleman (2003), Franco (2006) constructed explicit two-step hybrid methods of orders four, five and six for solving the second-order ODEs. Works on hybrid method for solving oscillatory problems have also been done by Samat et al. (2012), where they developed higher order hybrid method for solving oscillatory problems. Jikantoro et al. (2015) developed zero-dissipative trigonometrically fitted hybrid method for numerical solution of oscillatory problems.

Lots of researches on block methods for solving second order ODEs have been done, for instance, the work of Fatunla (1995), in which he developed block multistep method for directly solving special second order ODEs. Anake et al. (2012) have derived one-step implicit hybrid block method for the direct solution of general second order ODEs. Ismail et al. (2009) have derived explicit and implicit 3-point block multi-step methods using linear difference operator for solving special second order ODEs. Ramos et al. (2005) developed an optimized two-step hybrid block method for solving general second order initial-value problems. However, most of the block methods performed well for small interval of the problems. It is difficult to find block methods that solve efficiently the problems that are oscillatory in nature over a large integration interval.

Hence, we derive algebraic order conditions for block explicit hybrid method up to order five using the approach of B-series. Work on the B-series theory is based mostly on the work proposed by Coleman (2003) and Hairer et al. (2010). It has been shown that for oscillatory problems, phase-fitting or trigonometric-fitting of the methods gives more accurate numerical results (Hans Van de Vyver 2007; Jikantoro et al. 2015). Based on the order conditions, we develop two-point block explicit hybrid methods of order four and five. Then, we trigonometrically fit the methods so that they are suitable for solving highly oscillatory problems. Finally, we test the new methods using several large interval test problems to indicate that the methods are very efficient in solving oscillatory problems.

DERIVATION OF ORDER CONDITION FOR BLOCK HYBRID METHOD

The general form of an explicit hybrid method for solving the special second order ODEs can be written as

$$Y_i = (1 + C_i) y_n - c_i y_{n-1} + h^2 \sum_{j=1}^s a_{ij} f(x_n + c_j h, Y_j),$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^s b_i f(x_n + c_i h, Y_i),$$

where $i = 1, \dots, s$, and $i > j$. Further details on this method can be seen in Franco (2006). The coefficients; b_i , c_i , and can be represented in Butcher tableau as follows:

TABLE 1. The explicit hybrid methods

$$\begin{array}{c|c} c & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array},$$

where $\mathbf{A} = \begin{bmatrix} a_{11} & & \\ \vdots & \ddots & \\ a_{s1} & \cdots & a_{s,s-1} \end{bmatrix}$, $c = \begin{bmatrix} c_1 \\ \vdots \\ c_s \end{bmatrix}$, and $\mathbf{b}^T = \begin{bmatrix} b_1 \\ \vdots \\ b_s \end{bmatrix}$.

B-series and associated trees:

Rooted tree is a simple combinatorial graph with the property of being connected, with no cycle and having a designated vertex called the root (Butcher 2008). It is constructed from the derivatives of the differential equations under consideration. In this paper, the rooted trees and B-series theory for special second order ODEs developed in (Coleman 2003) are used and adapted for derivation of order conditions of the block hybrid methods.

To derive the order conditions for block explicit hybrid methods (BEHM), first, we shall consider them as a single-step method, without loss of generality, as

$$v_n = v_{n-1} + h\phi(v_{n-1}, h) \tag{2}$$

where v_n is a vector of numerical approximate solutions whose initial value v_0 is given by some starting procedures. Let $F_n := y_{n+2}$ so that

$$F_{n+2} = 2F_n - F_{n-2} + h^2 \sum_{i=1}^s b_i f(x_n + c_i h, Y_i). \tag{3}$$

Letting

$$G_n := \frac{F_{n+2} - F_n}{2h}, \text{ then } F_n = F_{n-1} + hG_{n-1}. \tag{4}$$

Therefore, (3) becomes

$$G_n = G_{n-1} + h \sum_{i=1}^s b_i f(x_n + c_i h, Y_i) \tag{5}$$

The system of (4) and (5) can be written as (2) with

$$v_n = \begin{pmatrix} F_n \\ G_n \end{pmatrix} \text{ and } \phi(v_{n-1}, h) = \begin{pmatrix} G_n \\ \sum_{i=1}^s b_i f(x_n + c_i h, Y_i) \end{pmatrix}.$$

The vector v_n is an approximation of vector $v_n = u(x_n, h)$, where

$$u(x, h) = \begin{pmatrix} F(x) \\ \frac{F(x+h) - F(x)}{h} \end{pmatrix}. \tag{6}$$

The local truncation error of BEHM at x^n is given by

$$l_n = u_n - u_{n-1} - h\phi(u_{n-1}, h). \tag{7}$$

with

$$\phi(u_{n-1}, h) = \begin{pmatrix} F(x_{n-1}) \\ \sum_{i=1}^s b_i f(x_n + c_i h, Y_i) \end{pmatrix}. \tag{8}$$

Theorem 1: For exact starting values, the block hybrid method is convergent of order p if and only if, for trees $t \in T_2$,

$$\sum_{i=1}^s b_i \psi_i''(t) = 2^{\rho(t)} + (-2)^{\rho(t)}. \tag{9}$$

For $\rho(t) \leq p + 1$ but not for some trees of order $p + 2$ where $t, T_2, \rho(t)$ and $\psi''(t)$ are trees, set of trees, order of tree and coefficient of B_2 series respectively, for further details see (Coleman 2003).

Proof: From (6) and (8) it is clear that the first component of is equal to zero and the second component is,

$$\frac{1}{h} \left[B(2^{\rho(t)}, y(x_n)) - 2y(x_n) + B((-2)^{\rho(t)}, y(x_n)) - \sum_{i=1}^s b_i (\psi_i''(t), y(x_n)) \right].$$

B denotes B-series. The block hybrid method is therefore said to be of order p if p is the largest integer such that

$$l_n = O(h^{p+1}), \tag{10}$$

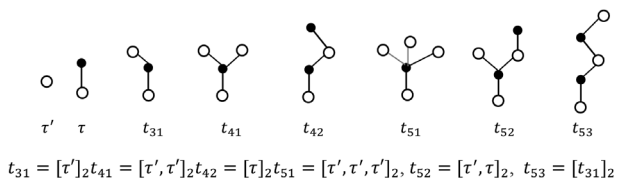
$\forall n \geq 0$. Equation (9) is satisfied $\Leftrightarrow \sum_{i=1}^s b_i \psi_i''(t) = 2^{\rho(t)} + (-2)^{\rho(t)}$ for $\rho(t) \leq p + 1$.

Hence, the proof. Where $\rho(t) \geq 2$

$$\psi_j''(t) = \rho(t)(\rho(t) - 1) \prod_{i=1}^m \psi_j(t), \tag{11}$$

$$\psi_i(t) = (-1)^{\rho(t)} c_i + \sum_j a_{ij} \psi_j''(t), \tag{12}$$

As in Coleman (2003), the trees associated with the hybrid methods are:



The subscript 2 is to indicate that the process of accumulating the trees adds two new vertices.

Remarks: For convenience, it is better to restrict attention to system of autonomous differential equations, $y'' = f(y)$. The first and higher derivatives of y can be represented using rooted trees. The rooted trees consist of two type of vertices, the white circle represents y' and the black dot represents f . A line leaving a vertex indicates partial derivative with respect to y if the end vertex is a circle and with respect to y' if the end vertex is a black dot.

And $\psi(\theta) = 1, \psi(\tau') = c_i$. θ is an empty tree, τ' is tree of order one and τ is tree of order two. The algebraic order conditions can now be obtained as follows:

Order condition for tree of order $\rho(t) = 1$

Order condition for tree of order one is trivial, see equation (11). Hence, the minimum order of the method is greater than one.

Order condition for tree of order $\rho(t) = 2$

From (11), $\psi_j''(\tau) = 2\psi_j(\theta) = 2$, then from (9) we have $\sum b_i = 4$ and from (12), we have $\psi_i(\tau) = 2\sum a_{ij} - c_i$.

Order condition for tree of order $\rho(t) = 3$

Equation(11) gives $\psi_j''(t_{31}) = 2\psi_j(\tau') = 6c_i$, which results in $\sum b_i c_i = 0$.

From (12), we get $\psi_i(t_{31}) = 6\sum a_{ij} c_j + c_i$.

Order condition for tree of order $\rho(t) = 4$

There are two trees with such order in T_2 , they are $t_{41} = [\tau', \tau']$ and $t_{42} = [\tau]_2$.

From (11), we have $\psi_j''(t_{41}) = 12\psi_j(\tau')\psi_j(\tau') = 12c_i^2$, then from (9) we have, $\sum b_i c_i^2 = \frac{8}{3}$.

and from (11), the following equations are obtained, $\psi_j''(t_{42}) = 12\psi_j(\tau) = 12(2\sum a_{ij} - c_i)$. Followed by $24\sum b_i a_{ij} - 12\sum b_i c_i = 32$, which implies $\sum b_i a_{ij} = \frac{4}{3}$, from (9)

Order condition for tree of order $\rho(t) = 5$

From (11), we obtained $\psi_j''(t_{51}) = 20c_i^3$.

From (9) we have $\sum b_i c_i^3 = 0, \psi_j''(t_{52}) = 20c_i(2\sum a_{ij} - c_i)$, giving $\sum b_i c_i a_{ij} = \frac{4}{3}$, and $\psi_j''(t_{53}) = 20\psi(t_{31}) = 20(6\sum a_{ij} c_j + c_i)$.

From (9), we obtain $\sum b_i a_{ij} c_j = 0$.

Then $24\sum b_i a_{ij} - 12\sum b_i c_i = 32$, giving $\sum b_i a_{ij} = \frac{4}{3}$.

Thus, we obtain the order condition for block hybrid method up to order 5 as:

TABLE 2. Order conditions of block hybrid method

Order 2:	$\sum_{i=1}^s b_i = 4$ (13)	Order 5:	$\sum_{i=1}^s b_i c_i^3 = 0$ (17)
Order 3:	$\sum_{i=1}^s b_i c_i = 0$ (14)		$\sum_{i=1}^s b_i c_i a_{ij} = \frac{4}{3}$ (18)
Order 4:	$\sum_{i=1}^s b_i c_i^2 = \frac{8}{3}$ (15)		$\sum_{i=1}^s b_i a_{ij} c_j = 0$ (19)
	$\sum_{i=1}^s b_i a_{ij} = \frac{4}{3}$ (16)		

CONSTRUCTION OF FOUR-STAGE FIFTH-ORDER TRIGONOMETRICALLY FITTED BLOCK EXPLICIT HYBRID METHOD

The general formula for block explicit hybrid method (BEHM) can be written as,

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^s a_{ij} f(x_n + c_j h, Y_j), \tag{20}$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^s {}^{(1)}b_i f(x_n + c_i h, Y_i), \tag{21}$$

$$y_{n+2} = 2y_n - y_{n-2} + h^2 \sum_{i=1}^s {}^{(2)}b_i f(x_n + c_i h, Y_i), \tag{22}$$

where the superscript on the left of b_i indicates the first or the second point. The four-stage fifth-order explicit hybrid method in (Franco 2006) is used as the method at the first point. The coefficients of BEHM4(5) are given in Table 3.

TABLE 3. The coefficient of four-stage fifth-order block explicit hybrid method (BEHM4 (5))

-1				
0				
$\frac{63}{100}$	$\frac{126651}{2000000}$	$\frac{900249}{2000000}$		
$-\frac{23}{37}$	$-\frac{43347640}{916464729}$	$-\frac{4864523}{50602347}$	$\frac{213026000}{8248182561}$	
	$\frac{31}{13692}$	$\frac{1675}{2898}$	$\frac{10000000}{47555739}$	$\frac{1874161}{8947092}$
	${}^{(2)}b_1$	${}^{(2)}b_2$	${}^{(2)}b_3$	${}^{(2)}b_4$

Solving (13) - (19) simultaneously, we obtain the values of ${}^{(2)}b_i$ as

$${}^{(2)}b_1 = \frac{124}{3423}, \quad {}^{(2)}b_2 = -\frac{3988}{1449},$$

$${}^{(2)}b_3 = \frac{160000000}{47555739}, {}^{(2)}b_4 = \frac{7496644}{2236773}$$

Requiring that the internal stage (20)–(22) to integrate exactly the linear combination of $\{\sin(vt), \cos(vt)\}$ for $v \in R$, the following equations are obtained:

$$\cos(c_3H) = 1 + c_3 - c_3 \cos(H) - H^2\{a_{31} \cos(H) + a_{32}\}, \quad (23)$$

$$\sin(c_3H) = c_3 \sin(H) + H_2\{a_{31} \sin(H)\}, \quad (24)$$

$$\begin{aligned} \cos(c_4H) = 1 + c_4 - c_4 \cos(H) - H^2\{a_{41} \cos(H) \\ + a_{42} + a_{43} \cos(c_3H)\}, \end{aligned} \quad (25)$$

$$\sin(c_4H) = c_4 \sin(H) + H^2\{a_{41} \sin(H) - a_{43} \sin(c_3H)\}, \quad (26)$$

$$\begin{aligned} 2\cos(H) = 2 - H^2\{{}^{(1)}b_1 \cos(H) + {}^{(1)}b_2 + \\ {}^{(1)}b_3 \cos(c_3H) + {}^{(1)}b_4 \cos(c_4H)\}, \end{aligned} \quad (27)$$

$${}^{(1)}b_1 \sin(H) = {}^{(1)}b_3 \sin(c_3H) + {}^{(1)}b_4 \sin(c_4H). \quad (28)$$

$$\begin{aligned} 2\cos(2H) = 2 - H^2\{{}^{(2)}b_1 \cos(H) + {}^{(2)}b_2 + \\ {}^{(2)}b_3 \cos(c_3H) + {}^{(2)}b_4 \cos(c_4H)\}, \end{aligned} \quad (29)$$

$${}^{(2)}b_1 \sin(H) = {}^{(2)}b_3 \sin(c_3H) + {}^{(2)}b_4 \sin(c_4H). \quad (30)$$

Solving (23), (24), (25) and (26) and letting $c_3 = \frac{63}{100}, c_4 = -\frac{23}{37}$, and $a_{43} = \frac{213026000}{8248182561}$; we obtained a_{31}, a_{32}, a_{41} and a_{42} in terms of H and to avoid heavy cancellation in the computation, the coefficient can be represented in Taylor series forms as;

$$\begin{aligned} a_{31} = & \frac{126651}{2000000} + \frac{2452512181}{400000000000} H^2 + \\ & \frac{146724514303373}{24000000000000000} H^4 + O(H^6), \\ a_{32} = & \frac{900249}{2000000} + \frac{2912905681}{400000000000} H^2 + \\ & \frac{18661530866927}{24000000000000000} H^4 + O(H^6), \\ a_{41} = & -\frac{43347640}{916464729} + \frac{2841353921}{624095613000} H^2 - \\ & \frac{235553288844846957}{512632136518200000000} H^4 + O(H^6), \\ a_{42} = & -\frac{4864523}{50602347} + \frac{3562448771}{624095613000} H^2 - \end{aligned}$$

$$\frac{409514991452584543}{512632136518200000000} H^4 + O(H^6),$$

Solving (27), (28), (29) and (30) with $c_3 = \frac{63}{100}, c_4 = -\frac{23}{37}$, we obtain the values of ${}^{(1)}b_i$ and ${}^{(2)}b_i$ in terms of H and can be represented in Taylor series as;

$$\begin{aligned} {}^{(1)}b_1 = & \frac{31}{13692} + \frac{31}{456400} H^2 + \frac{3662497511}{1686991320000000} H^4 \\ & + O(H^5), \\ {}^{(1)}b_2 = & \frac{1675}{2898} + \frac{31}{289800} H^2 - \frac{159219939391}{2499438060000000} H^4 \\ & + O(H^5), \\ {}^{(1)}b_3 = & \frac{10000000}{47555739} - \frac{1550}{47555739} H^2 + \frac{529824745097}{16406159286132000} \\ & H^4 + O(H^5), \\ {}^{(1)}b_4 = & \frac{1874161}{8947092} - \frac{42439}{298236400} H^2 + \frac{23542761989}{805238280000000} \\ & H^4 + O(H^5), \\ {}^{(2)}b_1 = & \frac{124}{3423} - \frac{31}{12225} H^2 + \frac{582976979}{452796750000} H^4 + O(H^5), \\ {}^{(2)}b_2 = & -\frac{3988}{1449} + \frac{8174}{12075} H^2 - \frac{608781814391}{156214878750000} H^4 \\ & + O(H^5), \\ {}^{(2)}b_3 = & \frac{160000000}{47555739} - \frac{485600}{1441083} H^2 + \frac{122001214019}{6290705247750} H^4 \\ & + O(H^5), \\ {}^{(2)}b_4 = & \frac{7496644}{2236773} - \frac{245051}{726225} H^2 + \frac{6851426568923}{352291747500000} H^4 \\ & + O(H^5), \end{aligned}$$

The new method is denoted as four-stage fifth-order trigonometrically fitted block explicit hybrid method (TF-BEHM4(5)).

NUMERICAL RESULTS

In this section, we use TF-BEHM4(5) for solving special second order ordinary differential equations (using C-Programming language) over a very large interval to show the efficiency of the method for integrating oscillatory problems. The method is then compared with other existing methods using a measure of the accuracy,

Absolute error which is defined by: *Absolute error* = $\max \{ \|y(x_n) - y_n\| \}$, where $y(x_n)$ is the exact solution and y_n is the computed solution.

Usually, a general purpose methods are developed without taking into account errors due to phase-lag and dissipation, as a result, they cannot integrate oscillatory problems accurately over a large interval. Therefore, to confirm that a method derived for solving oscillatory problems is effective, there is a need to integrate the problems over a large interval.

The linear and non-linear test problems are listed below:

Problem 1 (Inhomogeneous system by Lambert and Watson (1976)).

$$\frac{d^2 y_1(x)}{dx^2} = v^2 y_1(x) + v^2 f(x) + f''(x), y_1(0) = a + f(0),$$

$$y_1'(0) = f'(0),$$

$$\frac{d^2 y_2(x)}{dx^2} = v^2 y_2(x) + v^2 f(x) + f''(x), y_2(0) = f(0),$$

$$y_2'(0) = va + f'(0),$$

where $f(x) = e^{-0.05x}$, and $a = 0.1$. Exact solution is $y_1(x) = a \cos(vx) + f(x)$ and $y_2(x) = a \sin(vx) + f(x)$. The fitted frequency is $v = 20$.

Problem 2 (Homogeneous equation in Chakravarti and Worland (1971)).

$$y''(x) = -y(x), y(x) = 0, y'(0) = 1.$$

Exact solution is $y(x) = \sin(x)$. The fitted frequency is $v = 1$.

Problem 3 (Homogenous given in Basem et al. (2006)).

$$y''(x) = -64y(x), y(0) = 1/4, y'(0) = -1/2.$$

Exact solution is $y = \sqrt{17}/16 \sin(8x + \theta)$, $\theta = \pi - \tan^{-1}(4)$. The fitted frequency is $v = 8$.

Problem 4 (Inhomogeneous equation studied in Papadopoulos et al. (2009)).

$$y''(x) = -v^2 y(x) + (v^2 - 1) \sin(x), y(0) = 1, y'(0) = v + 1.$$

Exact solution is $y(x) = \cos(vx) + \sin(vx) + \sin(x)$. The fitted frequency is $v = 10$.

Problem 5 (Two-Body problem studied by Papadopoulos et al. (2009)).

$$y''_1(x) = \frac{-y_1(x)}{r^3}, y''_2 = \frac{-y_2(x)}{r^3}, \text{ where } r = \sqrt{y_1^2 + y_2^2}, y_1(0) = 1, y'_2(0) = 0, y_2(0) = 0, y'_1(0) = 1$$

Exact solution is $y_1(x) = \cos(x)$ and $y_2(x) = \sin(x)$. The fitted frequency is $v = 1$.

Problem 6 (A nonlinear orbital problem studied by Simos (2012)).

$$y''_1(x) + v^2 y_1(x) = \frac{2y_1(x)y_2(x) - \sin(2vx)}{(y_1(x)^2 + y_2(x)^2)^{3/2}},$$

$$y''_2(x) + v^2 y_2 = \frac{y_1(x)^2 - y_2(x)^2 - \cos(2vx)}{(y_1(x)^2 + y_2(x)^2)^{3/2}},$$

where $y_1(0) = 1, y_1'(0) = 0, y_2(0) = 0, y_2'(0) = v$.

Exact solution is $y_1(x) = \cos(vx)$ and $y_2(x) = \sin(vx)$. The fitted frequency is $v = 10$.

The following notations are used in Figures 1-6:

TF-BEHM4(5) : A four-stage fifth-order trigonometrically-fitted block explicit hybrid method developed in this paper.

EHM4(5) : A four-stage fifth-order explicit hybrid method by Franco (2006).

RKN4(5) : Explicit RKN method four-stage fifth-order by Hairer et al. (2010).

PF-RKN4(4) : Phase-fitted explicit Runge-Kutta Nyström method four-stage fourth-order by Papadopoulos et al. (2009).

IRKN(5) : A five-stage improved Runge-Kutta Nyström method by Rabiei et al. (2012).

MPAF-RKN4(4) : A phase-fitted and amplification fitted Runge-kutta Nystrm method by Hans Van de Vyver (2007).

The following algorithm is used in the implementation of the method.

Algorithm: To use TF-BEHM4(5) for solving $y'' = f(x, y)$ and to calculate the maximum absolute error.

Set value of v and variables coefficient for TF-BEHM4(5) method. ($v =$ fitted-frequency);

Set the maximum absolute error as $maxError = 0$;

Initialize the values of h, END, x_0, y_0 , and ($h =$ step size, $END =$ end of interval);

Calculate x_1, x_2, y_1 and y_2 .

($x_1 = x_0 + h, x_2 = x_1 + h$. Method such as Runge-Kutta or Euler is used to initialize y_1 and y_2).

For i to END

I. calculate y_3 and y_4 using the formula:

$$y_3 = (1 + c_3)y_2 - c_3y_1 + h^2(a_{31}f_1 + a_{32}f_2),$$

$$y_4 = (1 + c_4)y_2 - c_4y_1 + h^2(a_{41}f_1 + a_{42}f_2 + a_{43}f_3),$$

II. calculate f_1, f_2, f_3 , and f_4

$$(y'' = f(x, y))$$

III. calculate y_{n+1} and y_{n+2} using the formula:

$$y_{n+1} = 2y_2 - y_1 + h^2({}^{(1)}b_1f_1 + {}^{(1)}b_2f_2 + {}^{(1)}b_3f_3 + {}^{(1)}b_4f_4).$$

$$y_{n+2} = 2y_2 - y_0 + h^2({}^{(2)}b_1f_1 + {}^{(2)}b_2f_2 + {}^{(2)}b_3f_3 + {}^{(2)}b_4f_4).$$

(y_{n+1} is the first point and is the second point of the block method)

IV. calculate $max1 = \|y_{n+1} - exact\ solution\|$ and $max2 = \|y_{n+2} - exact\ solution\|$

V. if ($max2 > max1$)
 then if ($max2 > maxError$)
 then $maxError = max2$
 else $maxError = maxError$
 end if
 else ($max1 > max2$)
 then $maxError = max1$
 else $maxError = maxError$
 end if;

Set $i = i + 2h, y_0 = y_2, y_1 = y_{n+1}, y_2 = y_{n+2}, f_0 = f_2,$
 $x_0 = x_2, x_1 = x_2 + h, x_2 = x_1 + h.$
 $i < END$ go to step 5. end for ($i > END$);
 Return $maxError$. end of Algorithm.

The graphs of the logarithm of the absolute maximum errors are plotted against the CPU time taken (in seconds) for all the tested problems:

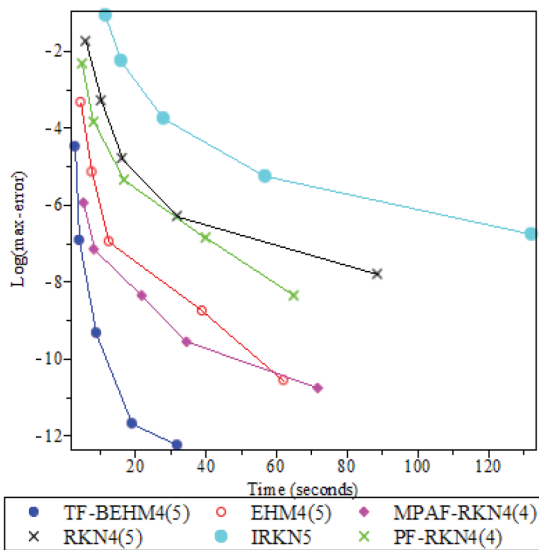


FIGURE 1. The efficiency curves for problem 1 for $h = \frac{0.5}{2^i}$, for $i = 1, \dots, 5$

DISCUSSION AND CONCLUSION

The efficiency curves are presented where the logarithms of the maximum global errors are plotted against the computational time in seconds. In analyzing the numerical results, methods of the same order and stage are compared. Figures 1-6 show the efficiency curves of the proposed fifth order trigonometrically-fitted method (TF-BEHM4(5)) and those of the existing methods, EHM4(5), RKN4(5), IRKN(5) and PF-RKN4(4). It is clear that the new trigonometrically fitted block method gives the least error and the shortest time.

In this paper, we derived algebraic order conditions for two-point block hybrid method using B-series approach. Then the order conditions are used to construct fifth order two-point block method for solving special second order ODEs. It is then trigonometrically fitted so

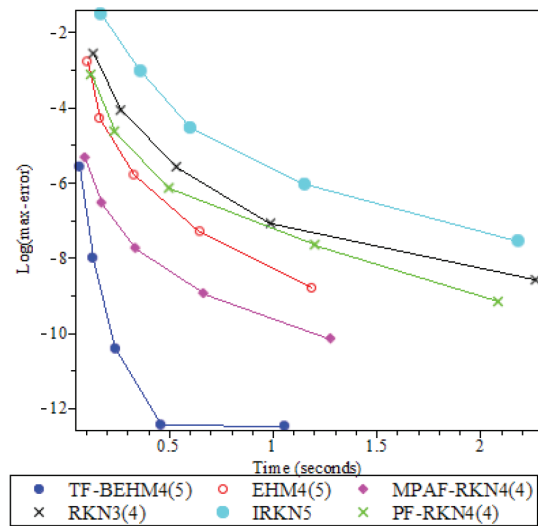


FIGURE 2. The efficiency curves for problem 2 for $h = \frac{0.125}{2^i}$, for $i = 3, \dots, 7$

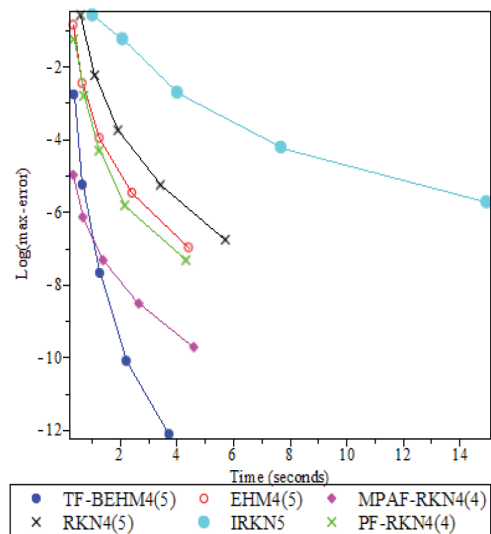


FIGURE 3. The efficiency curves for problem 3 for $h = \frac{1.0}{2^i}$, for $i = 4, \dots, 8$

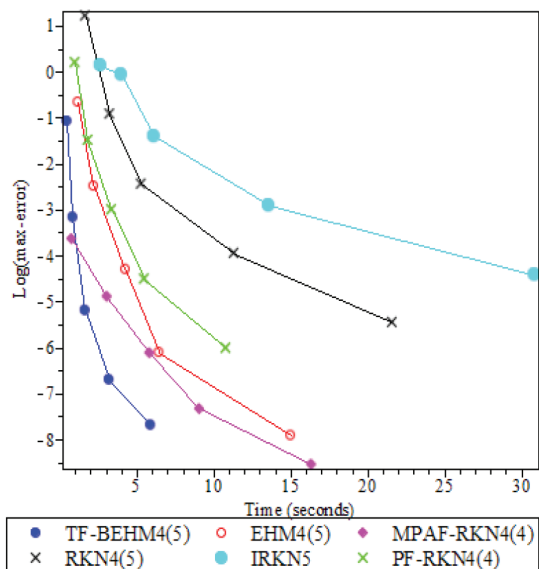


FIGURE 4. The efficiency curves for problem 4 for $h = \frac{0.125}{2^i}$, for $i = 1, \dots, 5$

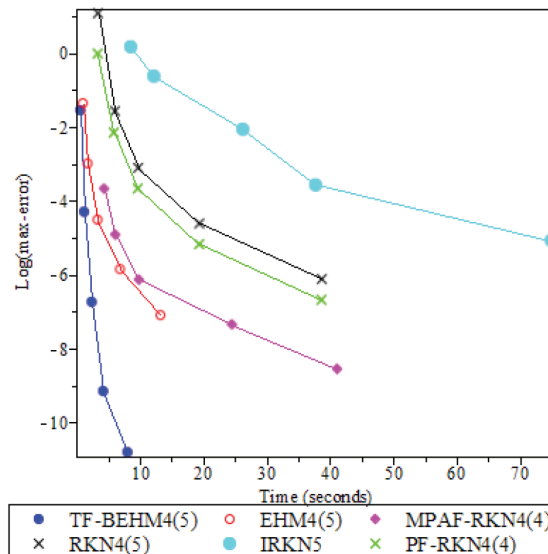


FIGURE 6. The efficiency curves for problem 6 for $h = \frac{0.125}{2^i}$, for $i = 0, \dots, 4$

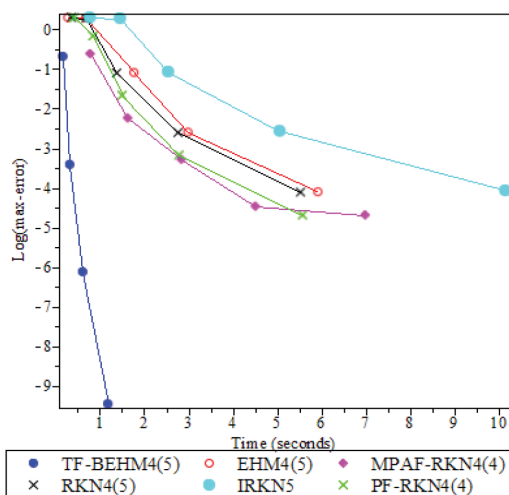


FIGURE 5. The efficiency curves for problem 5 For $h = \frac{0.25}{2^i}$, for $i = 0, \dots, 4$

that it is suitable for solving highly oscillatory problems. Efficiency curves presented in Figures 1-6 clearly showed that the trigonometrically fitted two-point block method (TF-BEHM4(5)) is more efficient than other existing methods in the literature. Though most of the existing methods were also constructed purposely for solving highly oscillatory problems that is, they are constructed with minimal phase-lag and dissipation. The proposed method performed better because the method takes into account the periodicity of the problems, hence less error are accumulated. The new method also calculate two-points at each time step, hence lesser time is required to do the computation, thus, it is faster and cheaper compared to the other methods.

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