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Constrained Interpolation using Rational Cubic Spline with Three Parameters

(Interpolasi Berkekangan menggunakan Splin Kubus dengan Tiga Parameter)

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ABSTRACT

The C^1 rational cubic spline function (cubic/quadratic) with three parameters is used to construct a constrained interpolating curve that lies below or above an arbitrary straight line or between two straight lines. The data dependent sufficient conditions for the rational cubic interpolant bounded by two straight lines are derived on one parameter, while the other two are free parameters that will be useful for shape modification. Some numerical results will be presented by using Mathematica software. Comparison with some existing schemes shows that the proposed scheme outperforms the existing schemes.

Keywords: Constrained interpolation; continuity; rational cubic spline

ABSTRAK

Fungsi splin kubus nisbah C^1 dengan tiga parameter digunakan untuk membentuk satu lengkung berkekangan yang berada di bawah atau atas suatu garis lurus atau di antara dua garisan lurus. Syarat cukup bagi lengkung kubus nisbah terbatas di antara dua garisan lurus yang bergantung kepada data dihasilkan pada satu parameter dengan dua yang lain adalah parameter bebas yang berguna untuk perubahan bentuk. Beberapa hasil berangka akan ditunjuk dengan menggunakan perisian Mathematica. Perbandingan dengan beberapa skema sedia ada menunjukkan bahawa skema yang dicadangkan mengatasi skema sedia ada.

Kata kunci: Interpolasi berkekangan; keselanjaran; splin kubus nisbah

INTRODUCTION

Shape preserving interpolation is an important task in computer graphics visualization. For instance, if the given data is positive, then the resulting interpolating curves or surfaces must be able to retain the positivity of the data, hence any negativity or non-positivity is unacceptable. Meanwhile, constrained data interpolation has been used in designing a smooth curve to generate the smooth robot's path to avoid corners or polygonal objects (Meek et al. 2003). The problem statement in constrained data interpolation can be read as follows: Given the scalar or functional data points, construct the interpolating curve that lies below or above the given straight line, or that lies between two straight lines.

There are many works that have discussed the shape preserving interpolation and constrained data interpolation. For instance, Abbas et al. (2012) studied the constrained shape preserving using rational bi-cubic spline (cubic denominator) with three parameters. Duan et al. (2005) studied the constrained control of the interpolation curves. Their scheme only works for equally spaced data and it won't work if the first derivatives are pre-specified at the data points. Goodman et al. (1991) studied the constrained interpolation of data points that lie on one side of one or two more lines with G^2 continuity. Their schemes have been improved by Meek et al. (2003) in which they constructed G^1 interpolating curves of ordered planar data points lying on one side of polyline and the number of inflection points are as minimal as possible. Hussain and Hussain (2006) studied the constrained data interpolation using rational cubic spline with two parameters. Their schemes were later improved by Karim and Kong (2014). The constrained interpolating schemes of Karim and Kong (2014) outperform the schemes presented by Hussain and Hussain (2006) in the sense of the smoothness of the resulting interpolating curves and the existing of the free parameters which enable the user to alter the final shape of interpolating curves. Sarfraz et al. (2013a) developed a new curve interpolating scheme based on a C^1 piecewise rational cubic function with two parameter family. Sarfraz et al. (2015) studied the constrained data interpolation for data that lies below or above a straight line and between two straight lines. Based on their results, there are some drawbacks of their schemes especially in the development of constrained data interpolation for data that lie between two straight lines. For instance, it was noticed that for all the tested data sets given in their paper, the default cubic Hermite spline polynomial already preserves the shape of the given data sets. Furthermore, for constrained data interpolation that lies between two straight lines, their scheme only tested to the data lies above arbitrary straight line, but not for below arbitrary straight line. Consequently, their scheme is not suitable for constrained data interpolation and possibly may not produce the constrained interpolating curve that lies between two straight lines. Shaikh et al. (2011) investigated the constrained data interpolation using rational cubic spline with quadratic denominator without any free parameters to refine the final shape of the constrained interpolating curve. Bastian-Walther and Schmidt (1999) discussed the range restricted interpolation using Gregory's rational cubic spline with one parameter.

Motivated by the works of Meek et al. (2003) and Sarfraz et al. (2015), we shall improve the results of Sarfraz et al. (2015) to constrained data that lies between two arbitrary straight lines by using the rational cubic spline (cubic/quadratic) with three parameters of Karim and Kong (2014). The sufficient conditions for constrained data interpolation are derived on one parameter, meanwhile the other two are free parameters that can be used to alter the shape of the constrained interpolating curve. The proposed scheme does not require any knots insertion or trigonometric functions as was required in the schemes of Bashir and Ali (2013), Ibraheem et al. (2012) and Sarfraz et al. (2015).

MATERIALS AND METHODS

RATIONAL CUBIC SPLINE INTERPOLANT

This section is devoted to the definition of the rational cubic spline interpolation with three parameters initiated by Karim and Kong (2014). Given the data sets $\{(x_i, f_i), i = 0, 1, \ldots, n\}$, where $x_0 < x_1 < \ldots < x_n$. Let $h_i = x_{i+1} - x_i$, $\Delta_i = \frac{(f_{i+1} - f_i)}{h_i}$ and a local variable, $\theta = \frac{(x - x_i)}{h_i}$. For $x \in [x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, n - 1$, the rational cubic spline is defined by:

$$S(x) \equiv S_i(x) = \frac{P_i(\theta)}{Q_i(\theta)}, \tag{1}$$

where $0 \leq \theta \leq 1$ and

$$P_i(\theta) = \alpha_i f_i (1-\theta)^3 + A_1\theta(1-\theta)^2 + A_2\theta^2(1-\theta) + \beta_i f_{i+1}\theta^3,$$

$$Q_i(\theta) = (1-\theta)^2 \alpha_i + \theta(1-\theta)(2\alpha_i\beta_i + \gamma_i) + \theta^2\beta_i.$$

The rational cubic spline interpolation defined by (1) satisfies the following C^1 continuity:

$$S(x_i) = f_i, \quad S(x_{i+1}) = f_{i+1},$$

$$S_i^{(1)}(x_i) = d_i, \quad S_i^{(1)}(x_{i+1}) = d_{i+1}, \tag{2}$$

where $S_i^{(1)}(x)$ denotes derivative with respect to x and d_i denotes the derivative value which is given at the knots x_i, $i = 0, 1, 2, \ldots, n$.

Applying the C^1 condition given in (2), the unknowns A_i, $i = 1, 2$ are given as follows:

$$A_1 = (2\alpha_i\beta_i + \alpha_i + \gamma_i) f_i + \alpha_i h_i d_i,$$

$$A_2 = (2\alpha_i\beta_i + \beta_i + \gamma_i) f_{i+1} - \beta_i h_i d_{i+1}.$$

The parameters $\alpha_i, \beta_i > 0$, and $\gamma_i \geq 0$ (Karim & Kong 2014).

Some geometric observations with respect to the rational cubic spline interpolant defined by (1) are (Karim & Kong 2014):

(a) When $\alpha_i = \beta_i = 1$, and $\gamma_i = 0$, the rational cubic interpolant in (1) is a standard cubic Hermite spline polynomial given as:

$$S_i(x) = (1-\theta)^2(1+2\theta) f_i + \theta^2(3-2\theta) f_{i+1} + \theta(1-\theta)^2 d_i - \theta^2(1-\theta) d_{i+1}. \tag{3}$$

(b) When $\alpha_i \to 0$ and $\beta_i \to 0$, or $\gamma_i \to \infty$, the rational interpolant in (1) converges to the straight line given as:

$$\lim_{\alpha_i,\beta_i \to 0} S_i(x) = \lim_{\gamma_i \to \infty} S_i(x) = (1-\theta) f_i + \theta f_{i+1} \tag{4}$$

The rational cubic spline interpolant in (1) can be rewritten as

$$S_i(x) = (1-\theta) f_i + \theta f_{i+1} + \frac{h_i\theta(1-\theta)\left[\alpha_i(d_i - \Delta_i)(1-\theta) + \beta_i(\Delta_i - d_{i+1})\theta\right]}{Q_i(\theta)}. \tag{5}$$

DETERMINATION OF DERIVATIVES

For functional or scalar data sets, the derivative parameters need to be estimated through some numerical methods. A simple method is the arithmetic mean method (AMM). The formula for AMM is as follows (Delbourgo & Gregory 1985):

At the end points x_0 and x_n

$$d_0 = \Delta_0 + (\Delta_0 - \Delta_1)\left(\frac{h_0}{h_0 + h_1}\right), \tag{6}$$

$$d_n = \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})\left(\frac{h_{n-1}}{h_{n-1} + h_{n-2}}\right). \tag{7}$$

At the interior points, x_i, $i = 1, 2, \ldots, n - 1$, the values of d_i are given as

$$d_i = \frac{h_{i-1}\Delta_i + h_i\Delta_{i-1}}{h_{i-1} + h_i}. \tag{8}$$

CONSTRAINED DATA MODELING

In this section, the sufficient conditions for the interpolating curve that lies (a) above, (b) below any arbitrary straight line or (c) between two straight lines will be derived on one parameter with the remaining two parameters can be used to modify the final interpolating curves. Here we assume that the rational cubic interpolation takes three different interpolating curves with different shape parameters. For data lies above the given straight line, let the parameters be $\alpha_{i,*}, \beta_{i,*}$ and $\gamma_{i,*}$. Similarly for data lies below the given straight line, the parameters are α_i^*, β_i^*, and γ_i^* respectively. For the data lies between two straight line the parameters are α_i, β_i, and γ_i. To simplify the derivation, we assume that the free parameters satisfy $\alpha_i^* = \alpha_{i,*} = \alpha_i, \beta_i^* = \beta_{i,*} = \beta_i$, for $i = 0, 1, \ldots, n-1$. Thus, for constrained data modeling, there are three cases that can be considered.

Case I

Let us assume the data lie above any arbitrary straight line $y = m_1x + c_1$. We write the straight line into the parametric form $L_i(\theta) = a_i(1-\theta) + b_i\theta$, where $a_i = m_1x_i + c_1$ and $b_i = m_1x_{i+1} + c_1$ for $i = 0, 1, \ldots, n-1$. Karim and Kong (2014) have presented the following result for $S_i(x) > L_i(\theta), \forall_i = 0, 1, 2, \ldots, n-1$:

THEOREM 1 (Karim & Kong 2014). The piecewise rational cubic spline interpolant $S(x)$ preserves the shape of the data that lie above the given straight line $y = m_1x + c_1$ if in each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$, the parameter $\gamma_{i,*}$ has the value which satisfies the following sufficient condition:

$$\alpha_i, \beta_i > 0, i = 0, 1, \ldots, n-1,$$

$$\gamma_{i,*} > \max\left\{0, \frac{\alpha_i\left(-f_i - h_i d_i + b_i\right)}{f_i - a_i}, \frac{\beta_i\left(-f_{i+1} + h_i d_{i+1} + a_i\right)}{f_{i+1} - b_i}\right\}. \quad (9)$$

Case II

If the data lies below any arbitrary straight line $y = m_2x + c_2$, which has the parametric form $M_i(\theta) = k_i(1-\theta) + l_i\theta$ where $k_i = m_2x_i + c_2$ and for . The following theorem states the main result for $S_i(x)\ M_i(\theta), \forall_i = 0, 1, 2, \ldots, n-1$:

THEOREM 2 The piecewise rational cubic spline interpolant $S(x)$ preserves the shape of the data that lie below the given arbitrary straight line $y = m_2x + c_2$, if in each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$ the parameter value γ_i^* satisfies the following sufficient condition:

$$\gamma_i^* > \max\left\{0, \frac{\alpha_i\left(f_i + h_i d_i - l_i\right)}{k_i - f_i}, \frac{\beta_i\left(f_{i+1} - h_i d_{i+1} - k_i\right)}{l_i - f_{i+1}}\right\}. \quad (10)$$

Case III

Assume the data lie between any two arbitrary straight lines, $y_1 = m_1x + c_1$ and $y_2 = m_2x + c_2$, such that $L_i(\theta) < S_i(x) < M_i(\theta)$. The following theorem states the main result:

THEOREM 3 The piecewise rational cubic spline interpolant $S(x)$ lies between two straight lines if in each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$ the free parameter γ_i satisfies the following sufficient condition:

$$\gamma_i > \max\left\{0, \gamma_i^*, \gamma_{i,*}\right\}, \quad (11)$$

where $\gamma_{i,*}$ and γ_i^* are given in (9) and (10), respectively.

Proof

Case I

Assuming that, we are given a set of data (x_i, f_i), $i = 0, 1, \ldots, n$ lie above the straight line $y = mx + c$ such that

$$f_i > mx_i + c, i = 0, 1, \ldots, n. \quad (12)$$

From Karim and Kong (2014), the curve will lie above the straight $y = mx + c$, if the rational cubic spline $S(x)$ defined by (1) satisfies the following condition:

$$S(x) > mx + c, \forall x \in [x_0, x_n]. \quad (13)$$

Now, for each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$ the relation in (13) can be expressed as:

$$S_i(x) = \frac{P_i(\theta)}{Q_i(\theta)} > mx + c,\ i = 0,1,\ldots,n-1. \quad (14)$$

Condition (14) can be rewritten as follows:

$$S_i(x) = \frac{P_i(\theta)}{Q_i(\theta)} > L_i(\theta),\ i = 0,1,\ldots,n-1 \quad (15)$$

with $a_i = m_x + c$ and $b_i = mx_{i+1} + c$, $i = 0, 1, \ldots, n-1$. Simple derivation proves that for all $\alpha_i, \beta_i, \gamma_i > 0$, then $Q_i(\theta) > 0$, $i = 0, 1, \ldots, n-1$. Rearrange (15), the following inequality is obtained:

$$S_i(x) = \frac{P_i(\theta) - \left\{(1-\theta)a_i + \theta b_i\right\} \times Q_i(\theta)}{Q_i(\theta)} > 0, i = 0,1,\ldots,n-1. \quad (16)$$

By considering only the numerator in (16):
Let

$$U_i(\theta) = P_i(\theta) - \left\{(1-\theta)a_i + \theta b_i\right\} \times Q_i(\theta) = \sum_{j=0}^{3} c_{ij}(1-\theta)^{3-j}\theta^j, \quad (17)$$

with

$$c_{i0} = \alpha_i\left(f_i - a_i\right),$$
$$c_{i1} = \left(2\alpha_i\beta_i + \alpha_i + \gamma_{i,*}\right)f_i + \alpha_i h_i d_i - \left(2\alpha_i\beta_i + \gamma_{i,*}\right)a_i - \alpha_i b_i,$$
$$c_{i2} = \left(2\alpha_i\beta_i + \beta_i + \gamma_{i,*}\right)f_{i+1} - \beta_i h_i d_i - \left(2\alpha_i\beta_i + \gamma_{i,*}\right)b_i - a_i\beta_i,$$
$$c_{i3} = \beta_i\left(f_{i+1} - b_i\right).$$

Now, $U_i(\theta) > 0$ if all $c_{ij} > 0$. Clearly $c_{i0} > 0$, $c_{i3} > 0$, since $f_i - a_i > 0, f_{i+1} - b_i > 0$. Thus, the sufficient condition can be derived when $c_{il} > 0$ and $c_{i2} > 0$ respectively:

$$(2\alpha_i\beta_i + \alpha_i + \gamma_{i,*}) f_i + \alpha_i h_i d_i - (2\alpha_i\beta_i + \gamma_{i,*}) a_i - \alpha_i b_i > 0 \quad (18)$$

and

$$(2\alpha_i\beta_i + \beta_i + \gamma_{i,*}) f_{i+1} - \beta_i h_i d_{i+1} - (2\alpha_i\beta_i + \gamma_{i,*}) b_i - a_i\beta_i > 0. \quad (19)$$

Conditions (18) and (19) provide the following relations,

$$\gamma_{i,*} > \frac{\alpha_i\left(-f_i - h_i d_i + b_i\right)}{f_i - a_i} \quad (20)$$

and

$$\gamma_{i,*} > \frac{\beta_i\left(-f_{i+1} + h_i d_{i+1} + a_i\right)}{f_{i+1} - b_i}. \quad (21)$$

Combining both conditions (20) and (21) leads us to

$$\gamma_{i,*} > \max\left\{0, \frac{\alpha_i\left(-f_i - h_i d_i + b_i\right)}{f_i - a_i}, \frac{\beta_i\left(-f_{i+1} + h_i d_{i+1} + a_i\right)}{f_{i+1} - b_i}\right\}, \quad i = 0,1,\ldots,n-1. \quad (22)$$

For the computer implementation, the sufficient conditions in (22) can be written as follows:

$$\gamma_{i,*} = \upsilon_i + \max\left\{0, \frac{\alpha_i\left(-f_i - h_i d_i + b_i\right)}{f_i - a_i}, \frac{\beta_i\left(-f_{i+1} + h_i d_{i+1} + a_i\right)}{f_{i+1} - b_i}\right\}, \quad i = 0,1,\ldots,n-1, \quad (23)$$

where $\upsilon_i > 0$.

Case II

Similarly, if the given data set (x_i, f_i), $i = 0, 1, \ldots, n$ lie below the straight line $y = mx + c$, such that

$$f_i < k_i(1-\theta) + l_i\theta \quad (24)$$

Then $S_i(x) = \frac{P_i(\theta)}{Q_i(\theta)} < M_i(\theta)$, $i = 0,1,\ldots,n-1$ which can be written as:

$$S_i(x) = \frac{P_i(\theta) - \left\{(1-\theta)k_i + \theta l_i\right\} \times Q_i(\theta)}{Q_i(\theta)} < 0, \quad i = 0,1,\ldots,n-1. \quad (25)$$

Now, by considering the numerator in (25) we obtain:

$$W_i(\theta) = P_i(\theta) - \left\{(1-\theta)k_i + \theta l_i\right\} \times Q_i(\theta) = \sum_{j=0}^{3} e_{ij}(1-\theta)^{3-j}\theta^j, \quad (26)$$

where

$$e_{i0} = \alpha_i\left(f_i - a_i\right),$$
$$e_{i1} = \left(2\alpha_i\beta_i + \alpha_i + \gamma_i^*\right) f_i + \alpha_i h_i d_i - \left(2\alpha_i\beta_i + \gamma_i^*\right) k_i - \alpha_i l_i,$$
$$e_{i2} = \left(2\alpha_i\beta_i + \beta_i + \gamma_i^*\right) f_{i+1} - \beta_i h_i d_{i+1} - \left(2\alpha_i\beta_i + \gamma_i^*\right) l_i - k_i\beta_i,$$
$$e_{i3} = \beta_i\left(f_{i+1} - b_i\right).$$

Now, $W_i(\theta) < 0$ if all $e_{ij} < 0$. Clearly $e_{i0} < 0$ and $e_{i3} < 0$ since $f_i - a_i < 0$, $f_{i+1} - b_i < 0$. The sufficient condition is given as:

$$\left(2\alpha_i\beta_i + \alpha_i + \gamma_i^*\right) f_i + \alpha_i h_i d_i - \left(2\alpha_i\beta_i + \gamma_i^*\right) k_i - \alpha_i l_i < 0 \quad (27)$$

and

$$\left(2\alpha_i\beta_i + \beta_i + \gamma_i^*\right) f_{i+1} - \beta_i h_i d_{i+1} - \left(2\alpha_i\beta_i + \gamma_i^*\right) l_i - k_i\beta_i < 0. \quad (28)$$

Inequalities (27) and (28) give the following relations:

$$\gamma_i^* > \frac{\alpha_i\left(f_i + h_i d_i - l_i\right)}{k_i - f_i} \quad (29)$$

and

$$\gamma_i^* > \frac{\beta_i\left(f_{i+1} - h_i d_{i+1} - k_i\right)}{l_i - f_{i+1}}. \quad (30)$$

Inequalities (29) and (30) can be combined to form the sufficient condition for constrained interpolant that lies below any arbitrary straight line:

$$\gamma_i^* > \max\left\{0, \frac{\alpha_i\left(f_i + h_i d_i - b_i\right)}{a_i - f_i}, \frac{\beta_i\left(f_{i+1} - h_i d_{i+1} - a_i\right)}{b_i - f_{i+1}}\right\}, \quad i = 0,1,\ldots,n-1. \quad (31)$$

The sufficient conditions in (31) can be written as

$$\gamma_i^* = \lambda_i + \max\left\{0, \frac{\alpha_i\left(f_i + h_i d_i - b_i\right)}{a_i - f_i}, \frac{\beta_i\left(f_{i+1} - h_i d_{i+1} - a_i\right)}{b_i - f_{i+1}}\right\}, \quad i = 0,1,\ldots,n-1, \quad (32)$$

where $\lambda_i > 0$.

Case III

Assume that the data lies between any two arbitrary straight lines, $y_1 = m_1x + c_1$ and $y_2 = m_2x + c_1$, i.e. $L_i(\theta) < S_i(x) <$

$M_i(\theta)$. The sufficient condition trivially follows from Case I and Case II, respectively. The sufficient condition for this case is:

$$\gamma_i > \max\left\{0, \gamma_i^*, \gamma_{i,*}\right\} \quad (33)$$

The condition in (33) can be written as:

$$\gamma_i = \kappa_i + \max\left\{0, \gamma_{i,*}, \gamma_i^*\right\}, \ i = 0,1,...,n-1. \quad (34)$$

where $\kappa_i > 0$ and where $\gamma_{i,*}$ and γ_i^* are given in (23) and (32), respectively.

Remark Sufficient conditions in (23), (32) and (34) will be fulfilled for any positive value of the free parameters $\alpha_i, \beta_i > 0$. This is the main advantages of the proposed scheme as compared to that of Hussain and Hussain (2006) and Shaikh et al. (2011).

ALGORITHM 1 (Constrained data interpolation)

1. Input the $n + 1$ number of data points, and straight line equation, $y = mx + c$.
2. For $i = 0, 1, \ldots, n$, estimate d_i using the AMM formula.
3. For $i = 0, 1, \ldots, n - 1$
   - Choose free parameters $\alpha_i > 0, \beta_i > 0$
   - Calculate the shape parameter $\gamma_{i,*}, \gamma_i^*$ and γ_i using (23), (32) and (34), respectively.
4. For $i = 0, 1, \ldots, n - 1$
   Construct the piecewise constrained interpolating curve (below, above straight line or between two straight lines).

RESULTS AND DISCUSSION

In this section, we test the proposed scheme with various types of data sets including the comparison with the scheme of Sarfraz et al. (2015). All numerical results have been produced using Mathematica Version 9.

EXAMPLE 1. Data in Table 1 is taken from Sarfraz et al. (2015). The data lies above the straight line $y = 0.2x + 0.1$.

TABLE 1. Data from Sarfraz et al. (2015)

x_i	0	0.5	2	3	5
f_i	1.5	0.5	3	6	5

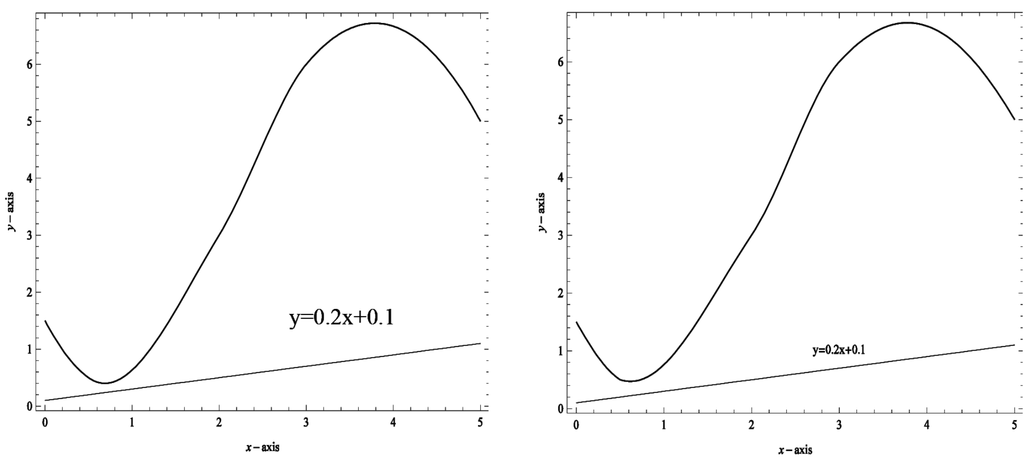
From Figure 1(a), it was noticed that even with cubic Hermite spline with $\alpha_i = \beta_i = 1$, $\gamma_i = 0$, the interpolating curve already lies above the given straight line. Thus, for this data set, the cubic Hermite spline is suitable for the shape preserving interpolation. Note that in general cubic Hermite spline is not shape preserving for all types of data set. Figure 1(b) shows the results using the scheme of Sarfraz et al. (2015).

EXAMPLE 2. Data in Table 2 is taken from Sarfraz et al. (2015). The data lies below the straight line $y = x + 2.2$.

TABLE 2. Data from Sarfraz et al. (2015)

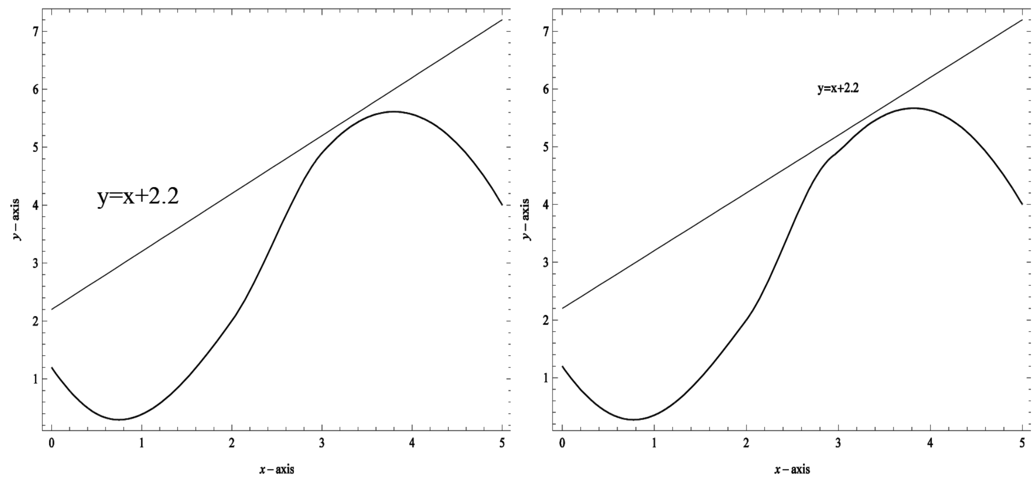
x_i	0	0.4	2	3	5
f_i	1.2	0.5	2	4.9	4

Figure 2(a) shows that the cubic Hermite spline interpolation also lies below the given straight line. Thus, cubic spline is suitable for interpolating the constrained data that lies below the given straight line. Figure 2(b) shows the results of the scheme of Sarfraz et al. (2015). Comparing Figures 1(a) and 1(b) with 2(a) and 2(b), the cubic spline interpolation gives the interpolating curve that is visually pleasing and smooth compared with the interpolating curve obtained by using the quadratic trigonometric spline of Sarfraz et al. (2015). For data sets



(a) (b)

FIGURE 1. (a) Default cubic spline polynomial $\alpha_i = \beta_i = 1$, $\gamma_i = 0$ and (b) shape preserving interpolation using Sarfraz et al. (2015) for the data in Table 1



(a) (b)

FIGURE 2. (a) Default cubic spline polynomial $\alpha_i = \beta_i = 1, \gamma_i = 0$ and (b) shape preserving polynomial using Sarfraz et al. (2015) for the data in Table 2

in Tables 1 and 2, we suggest that the user use the standard cubic Hermite spline polynomial without the need to apply any shape preserving scheme.

Thus, to show the capability of the proposed scheme, we consider the following examples.

EXAMPLE 3. Data in Table 3 lies above the straight line $y = 0.2x + 0.1$.

TABLE 3. Constrained data

x_i	0	0.5	1.6	2	3	5
f_i	1.5	0.5	0.6	3	6	5
d_i	-2.653	-1.347	4.424	5.143	1.833	-2.833

Figure 3(a) clearly shows that the default cubic Hermite spline interpolating curve lies below the given straight line on the sub-interval [0.5, 1.6]. This flaw has been recovered nicely by using the proposed scheme. Figure 3(b)-3(d) shows the constrained interpolating curve when $\alpha_i = \beta_i = 1$, $\alpha_i = \beta_i = 0.5$ and $\alpha_i = \beta_i = 2.5$, respectively. The shape preserving with $\alpha_i = \beta_i = 1$ gives visually pleasing interpolating curve.

EXAMPLE 4. Data in Table 4 is modified from Sarfraz et al. (2015). The data lies below the straight line $y = x + 2.1$.

TABLE 4. Constrained data below $y = x + 2.1$

x_i	0	0.4	2	3	4.2	5
f_i	1.2	0.5	2	4.9	5.8	4
d_i	-2.288	-1.213	2.145	1.923	-1.05	-3.45

Figure 4 shows that the resulting interpolating curves lie below the straight line. Figure 4(a) shows that the default cubic spline cannot preserve the shape of the given data sets especially on the interval (3.4, 4). Applying the main result in Theorem 2 produces the constrained interpolating curve which is visually pleasing. Figure 4(b)-4(d) shows the interpolating curve when $\alpha_i = \beta_i = 1$, $\alpha_i = \beta_i = 0.5$ and $\alpha_i = \beta_i = 2.5$, respectively. All three choices of free parameters give results which are visually pleasing while maintaining shape preservation.

EXAMPLE 5. The data in Table 3 is taken from Sarfraz et al. (2015). The data lies below the straight line $y = 6x + 30$ and above the straight line $y = x + 0.1$.

TABLE 5. Data from Sarfraz et al. (2015)

x_i	0	0.4	2	3	4	5.6	6
f_i	25	10	12	30	8	25	40

Similarly with the data set given in Tables 1 and 2, it was noticed that by using the default cubic spline interpolation with $\alpha_i = \beta_i = 1, \gamma_i = 0$, the interpolating curve lies between both straight lines. Thus, for this data set, the default cubic spline interpolant preserves the shape of the data. Sarfraz et al.'s (2015) scheme was only applied to the data that lies above a given straight line, since for this data set, their quadratic trigonometric spline polynomial lies below the given straight line. Obviously, their scheme was not tested on data that lies between two straight lines.

EXAMPLE 6. Data in Table 6 lies below the straight line $y = x + 2.1$ and above the straight line $y = 0.5x + 0.1$.

Figure 6 shows the examples of constrained data interpolation for data in Table 6. Figure 6(a) shows that the default cubic Hermite spline lies above the given straight line $y = x + 2.1$ on the interval (0.6,1.4) and lies below $y = 0.5x + 0.1$ on the interval (3.4,4). Applying only

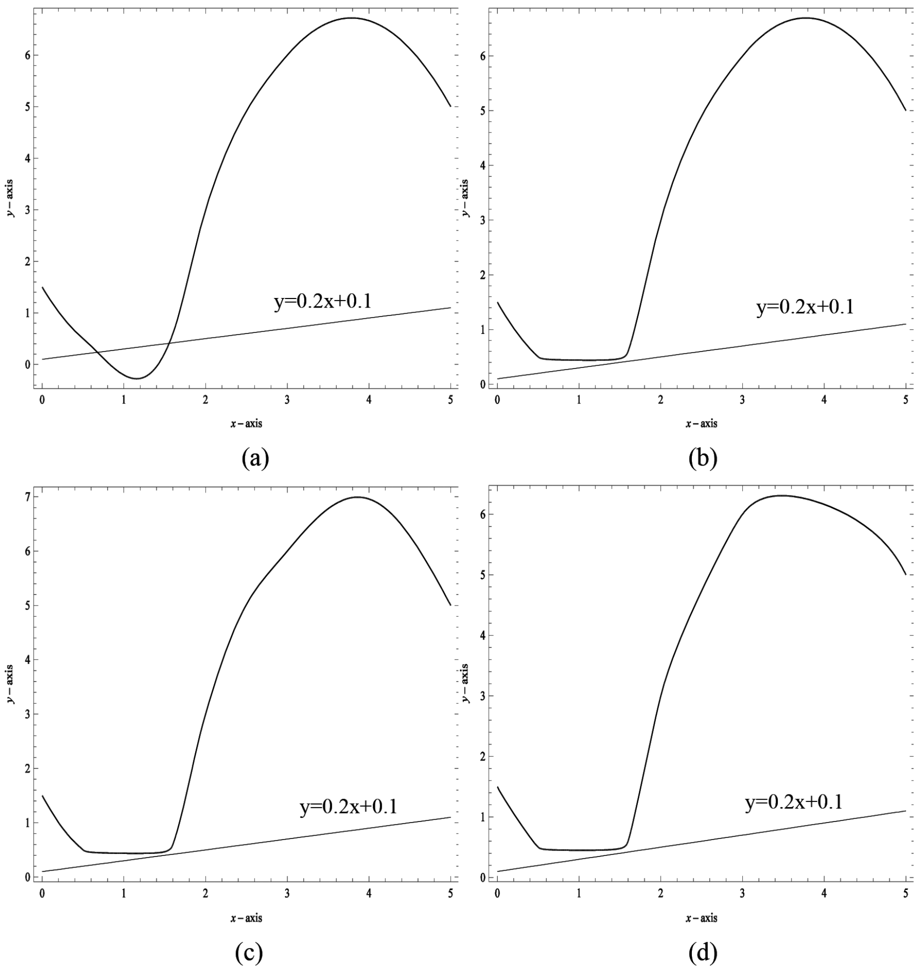


FIGURE 3. Constrained data interpolation for the data in Table 4, (a) Default cubic spline interpolation with $\alpha_i = \beta_i = 1$, $\gamma_i = 0$ and shape preserving interpolation using result in Theorem 1: (b) $\alpha_i = \beta_i = 1$, (c) $\alpha_i = \beta_i = 0.5$ and (d)

TABLE 6. Constrained data below $y = x + 2.1$ and above $y = 0.5x + 0.1$

x_i	0	0.4	2	3	4.2	5
f_i	1.2	0.5	2	4.9	5.8	4
d_i	-2.288	-1.213	2.145	1.923	-1.05	-3.45

Theorem 2 results in Figure 6(b) and similarly applying Theorem 1 yields Figure 6(c). Now applying the main result from Theorem 3 produces the interpolating curve that lies between the given straight lines as can be seen clearly in Figure 6(d). Thus, the sufficient condition in Theorem 3 can be used to construct the constrained interpolant for the data that lies between two straight lines. Figure 7 shows the constrained data interpolation between two straight lines with $\alpha_i = \beta_i = 0.5$, for $i = 0, 1, 2, 3, 4$.

Finally, we implement Sarfraz et al.'s (2015) scheme to the data listed in Table 6. We can see from Figure 8 that Sarfraz et al.'s (2015) scheme failed to preserve the constrained data between the two straight lines. This is due to the fact that their sufficient condition for data lying between two straight lines is not satisfied for all types of data sets. This can be interpreted as the resulting interpolant is negative. In contrast, with the proposed scheme in this study, our sufficient condition given in Theorem 3 is guaranteed to produce an interpolating curve which lies between two straight lines.

Based on all the examples considered as well as the ability to produce visually pleasing curve, we conclude that the proposed scheme works very well and outperforms the constrained data interpolation scheme constructed by Sarfraz et al. (2015) in terms of ease of use (not involving any trigonometric functions) and visually pleasing results. The conditions in Theorem 3 are sufficient for producing an interpolating curve that lies between two straight lines.

CONCLUSION

The rational cubic spline of Karim and Kong (2014) with three parameters has been used for constrained data interpolation with two constraint lines. The sufficient condition is derived on one parameter, i.e. γ_i, $\forall i = 0, 1, 2, \ldots, n-1$, meanwhile the other two parameters are free to

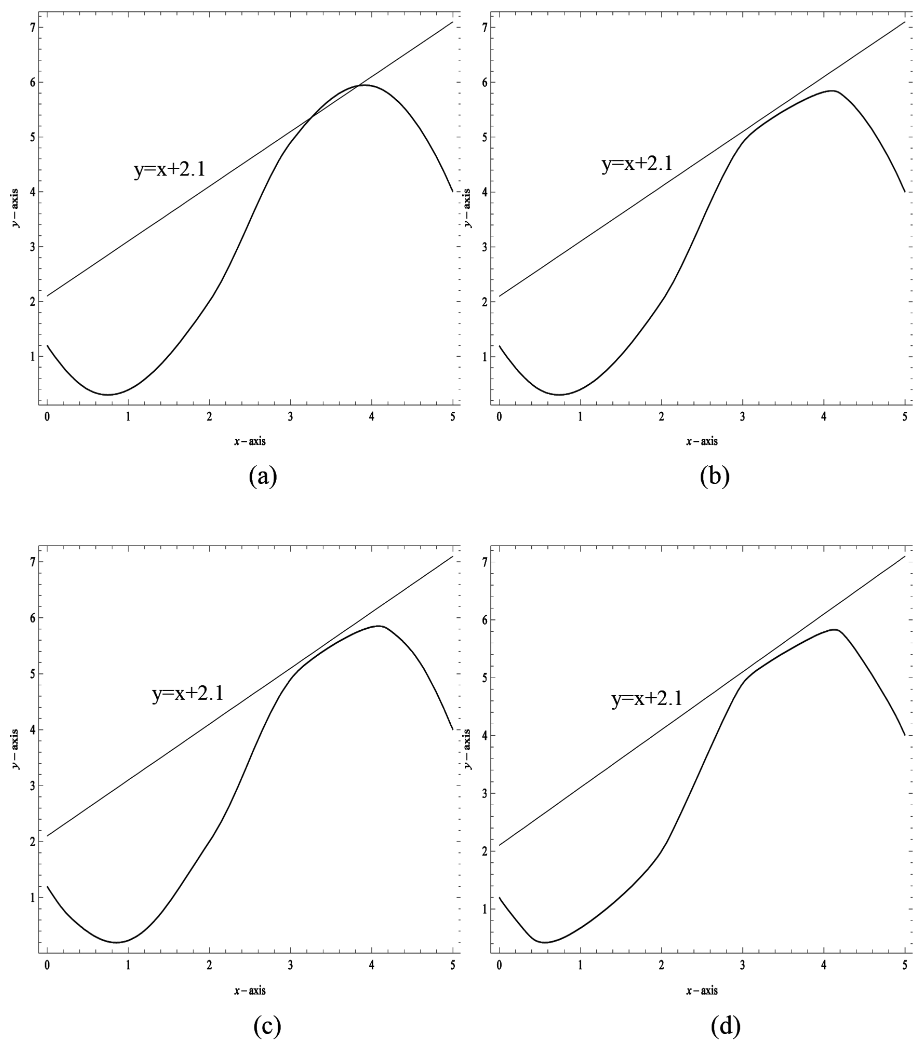


FIGURE 4. Constrained data interpolation for the data in Table 4, (a) Default cubic spline interpolation with $\alpha_i = \beta_i = 1, \gamma_i = 0$ and shape preserving interpolation using result in Theorem 2: (b) $\alpha_i = \beta_i = 1$, (c) $\alpha_i = \beta_i = 0.5$ and (d) $\alpha_i = \beta_i = 2.5$

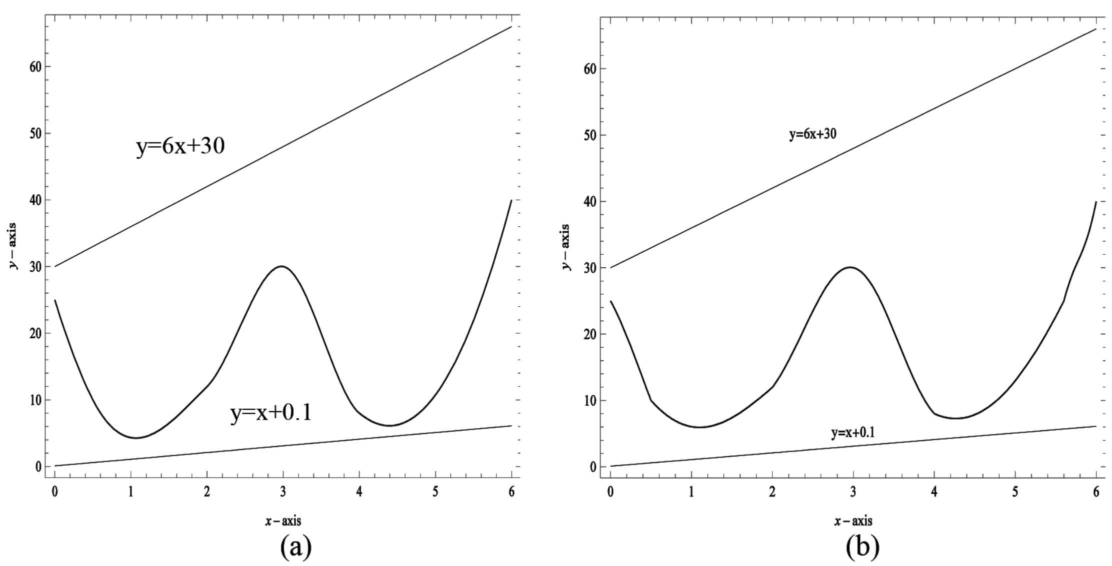


FIGURE 5. (a) Default cubic spline polynomial $\alpha_i = \beta_i = 1, \gamma_i = 0$ and (b) shape preserving curve using Sarfraz et al. (2015) for the data in Table 5

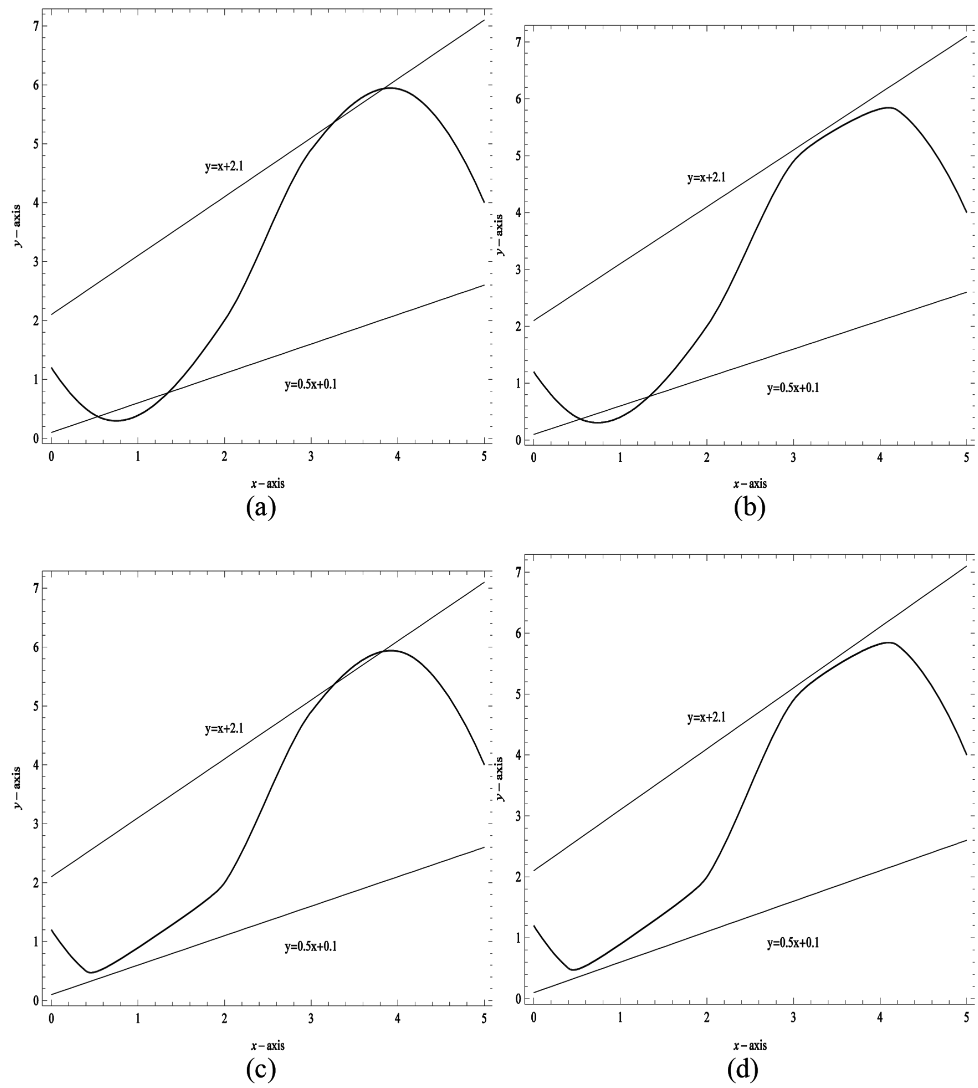


FIGURE 6. (a) Default cubic spline polynomial for the data in Table 6 and constrained data interpolation with $\alpha_i = \beta_i = 1$: (b) only below the given straight line $y = x + 2.1$, (c) only above the given straight line $y = 0.5x + 0.1$ and finally (d) between both straight lines by applying the main result stated in Theorem 3

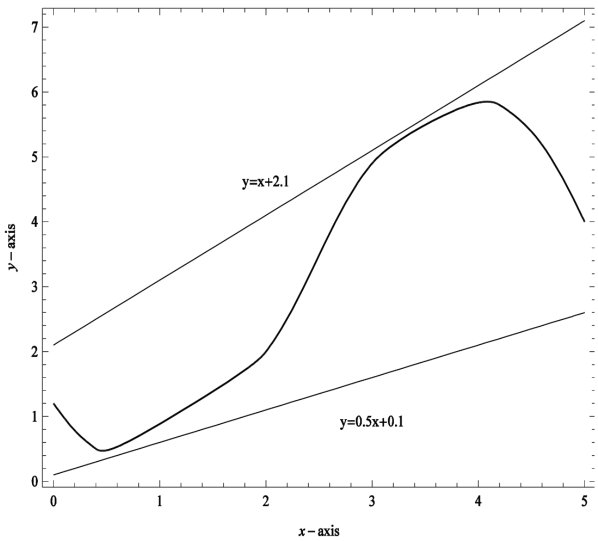


FIGURE 7. Constrained interpolating curve between two straight lines with $\alpha_i = \beta_i = 0.5$

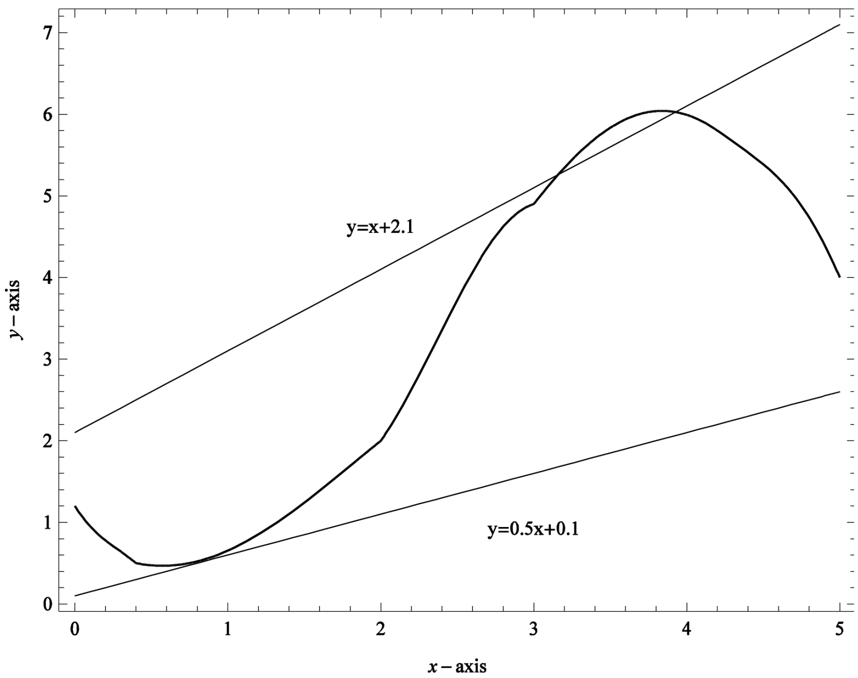


FIGURE 8. Constrained interpolating curve between two straight lines using Sarfraz et al. (2015)

be utilized in order to alter the shape of the interpolating curves. This is the main advantage over the schemes presented by Hussain and Hussain (2006), Sarfraz et al. (2015) and Shaikh et al. (2011). Based on all numerical examples, the proposed scheme can produce constrained interpolating curve that is bounded by two straight lines. In constrast, Sarfraz et al.'s (2015) scheme may give some interpolating curves that do not lie between two straight lines. Future works will focus on the constrained interpolation bounded by a straight line and quadratic polynomial. Applications to robot's path problems and medical image processing also seem possible using the proposed constrained data interpolation scheme by integrating genetic algorithm (GA) as discussed by Sarfraz et al. (2013b).

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