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A New Exponentiated Beta Burr Type X Distribution: Model, Theory, and Applications

(Taburan Beta Burr Jenis X Baru yang Dipertingkatkan: Model, Teori dan Aplikasi)

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ABSTRACT

In recent years, many attempts have been carried out to develop the Burr type X distribution, which is widely used in fitting lifetime data. These extended Burr type X distributions can model the hazard function in decreasing, increasing and bathtub shapes, except for unimodal. Hence, this paper aims to introduce a new continuous distribution, namely exponentiated beta Burr type X distribution, which provides greater flexibility in order to overcome the deficiency of the existing extended Burr type X distributions. We first present its density and cumulative function expressions. It is then followed by the mathematical properties of this new distribution, which include its limit behaviour, quantile function, moment, moment generating function, and order statistics. We use maximum likelihood approach to estimate the parameters and their performance is assessed via a simulation study with varying parameter values and sample sizes. Lastly, we use two real data sets to illustrate the performance and flexibility of the proposed distribution. The results show that the proposed distribution gives better fits in modelling lifetime data compared to its sub-models and some extended Burr type X distributions. Besides, it is very competitive and can be used as an alternative model to some nonnested models. In summary, the proposed distribution is very flexible and able to model various shaped hazard functions, including the increasing, decreasing, bathtub, and unimodal.

Keywords: Beta generalized; Burr type X; exponentiated; survival analysis; unimodal

ABSTRAK

Dalam beberapa tahun kebelakangan ini, banyak percubaan telah dijalankan untuk membangunkan taburan Burr jenis X yang digunakan secara meluas dalam model sepanjang hayat yang sesuai. Taburan lanjutan Burr jenis X ini boleh memodelkan fungsi *hazard* dalam bentuk menurun, meningkat dan *bathtub*, kecuali bagi unimod. Kertas ini bertujuan untuk memperkenalkan taburan berterusan baharu, iaitu taburan Burr jenis X beta eksponen, yang lebih keluwesan, bagi mengatasi kekurangan taburan lanjutan Burr jenis X sedia ada. Kami bermula dengan membentangkan ketumpatan dan ungkapan fungsi terkumpulnya. Ia kemudiannya diikuti dengan sifat matematik taburan baharu ini, yang merangkumi kelakuan hadnya, fungsi kuantil, momen, fungsi penjanaan momen dan statistik pesanan. Kami menggunakan pendekatan kemungkinan maksimum untuk menganggarkan parameter dan prestasinya dinilai melalui kajian simulasi dengan nilai parameter dan saiz sampel yang berbeza-beza. Akhir sekali, kami menggunakan dua set data sebenar untuk menggambarkan prestasi dan berkefleksibelan taburan yang dicadangkan. Keputusan menunjukkan bahawa taburan yang dicadangkan memberikan kesesuaian yang lebih baik dalam pemodelan data sepanjang hayat berbanding dengan sub-modelnya dan beberapa taburan lanjutan Burr jenis X. Selain itu, ia sangat bersaing dan boleh digunakan sebagai model alternatif kepada beberapa model tidak bersarang. Secara ringkasnya, taburan yang dicadangkan adalah sangat fleksibel dan boleh memodelkan pelbagai bentuk fungsi *hazard*, termasuk peningkatan, penurunan, *bathtub* dan unimod.

Kata kunci: Analisis kemandirian; beta teritlak; Burr jenis X; eksponen; unimod

INTRODUCTION

Statistical distributions are essential to statistical modelling in lifetime analysis. However, in some cases, the existing distributions could not fit the lifetime data well, especially for lifetime data with bathtub and unimodal shaped hazard function, which are common in survival analysis. Thus, exploring new forms of distributions with higher flexibility in modelling lifetime data becomes necessary. Given that, many attempts have been established to generate new distributions by extending the existing distributions. For instance, exponentiated Weibull Burr type XII (Abouelmagd, Hamed & Afify 2017), Kumaraswamy exponentiated Burr XII (Afify & Mead, 2017), beta-burr type V (Dikko, Aliyu & Alfa 2017), Weibull Burr XII (Afify et al. 2018), and generalized Marshall- Olkin extended Burr XII (Handique & Chakraborty 2018) distributions. These new distributions have higher flexibility compared to their baseline distribution.

Burr type X distribution (Burr 1942) is one of the famous distributions used in lifetime data analysis. Its initial form is a single-parameter distribution. Then, a two parameters scaled Burr type X distribution (BX) was introduced by Surles and Padgett (2001) by adding a scale parameter to the one-parameter Burr type X distribution. In literature, many attempts have been carried out by developing new families of Burr type X distribution. These include gamma Burr type X (GBX) (Khaleel et al. 2016), beta Burr type X (BBX) (Merovci et al. 2016), exponentiated generalised Burr type X (EGBX) (Khaleel et al. 2018), Weibull Burr type X (WBX) (Ibrahim et al. 2017), and beta Kumaraswamy Burr type X (Madaki, Bakar & Handique 2018) distributions. However, it is noticed that these extended Burr type X distributions only model the hazard functions with the shapes of decreasing, increasing and bathtub, but not for unimodal. The unimodal is a shape with a single peak and is commonly used in survival analysis. Since the flexibility of distribution is always the main concern in choosing a suitable model to fit the lifetime data, more attention is needed in developing the distribution with greater flexibility to improve the statistical findings.

In this study, we focus our discussion on the BBX distribution and look forward to enhancing its flexibility. Merovci et al. (2016) stated that with Burr type X distribution as the baseline distribution, the BBX distribution can be formed by implementing the beta-G (Eugene Lee & Famoye 2002). The probability density function (pdf) of BBX is given by

$$g(x,\alpha,\beta,\lambda,\theta) = \frac{2\theta\lambda^2 x e^{-(\lambda x)^2}}{B(\alpha,\beta)} \left(1-e^{-(\lambda x)^2}\right)^{\alpha\theta-1}\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta-1} \tag{1}$$

and the corresponding cumulative distribution function (cdf) becomes

$$G(x,\alpha,\beta,\lambda,\theta) = \frac{1}{B(\alpha,\beta)}\int_0^{\left(1-e^{-(\lambda x)^2}\right)^{\theta}} t^{\alpha-1}(1-t)^{\beta-1}\,dt = I_{\left(1-e^{-(\lambda x)^2}\right)^{\theta}}(\alpha,\beta), \tag{2}$$

where α, β, λ, $\theta > 0$ and $I_{\left(1-e^{-(\lambda x)^2}\right)^{\theta}}(\alpha,\beta)$, is the regularised beta function. The BBX hazard function then can be obtained by computing the ratio of pdf to survival function, such as

$$h(x,\alpha,\beta,\lambda,\theta) = \frac{2\theta\lambda^2 x e^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\alpha\theta-1}}{B(\alpha,\beta)\left[1-I_{\left(1-e^{-(\lambda x)^2}\right)^{\theta}}(\alpha,\beta)\right]} \left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta-1}. \tag{3}$$

It is worth mentioning that the hazard function in equation (3) only covers the shapes of increasing, decreasing and bathtub, but not for the unimodal shape. Thus, this study aims to generate a new exponentiated beta Burr type X distribution by adding a shape parameter to the BBX distribution using the exponentiated type of distribution proposed by Gupta, Gupta and Gupta (1998). By the additional shape parameter, the proposed distribution has greater flexibility than the BBX distribution by covering hazard functions in different shapes, including unimodal and bathtub. Besides, this distribution has several sub-models and thus can be used to model more comprehensive structural properties. We expect that the proposed distribution be superior to solve the deficiency of BBX.

The rest of the paper is outlined as follows. Next section presents the pdf, cdf, and hazard function of exponentiated beta Burr type X (EBBX) distribution. We explain the derivation of statistical properties and likelihood function in subsequent two sections. Besides, the limit behaviour of the pdf and cdf of EBBX distribution when x approaches zero and infinity are also presented in section that follows. We access the performance of EBBX distribution via simulation studies in the following section, while the implementation of EBBX distribution through two real data sets is illustrated next. Lastly, the paper ends with concluding remarks in the last section.

EXPONENTIATED BETA BURR TYPE X DISTRIBUTION

In this study, we introduce a new five-parameter distribution, namely exponentiated beta Burr type X (EBBX) distribution which is obtained by implementing the exponentiated type of distribution proposed by Gupta, Gupta and Gupta (1998) along with the BBX as the baseline distribution. By applying the exponentiated type of distribution, a shape parameter is added to the BBX distribution by taking exponent on equation (2). We may obtain the cdf of EBBX distribution such as

$$G(x,\alpha,\beta,\gamma,\lambda,\theta)=\frac{1}{[B(\alpha,\beta)]^{\gamma}}\left[\int_0^{\left(1-e^{-(\lambda x)^2}\right)^{\theta}} t^{\alpha-1}(1-t)^{\beta-1}dt\right]^{\gamma}$$

$$=\left[I_{\left(1-e^{-(\lambda x)^2}\right)^{\theta}}(\alpha,\beta)\right]^{\gamma} \quad (4)$$

and the corresponding pdf becomes

$$g(x,\alpha,\beta,\gamma,\lambda,\theta)=\frac{2\gamma\lambda^2\theta x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{\theta\alpha-1}}{[B(\alpha,\beta)]^{\gamma}}$$

$$[1-(1-e^{-(\lambda x)^2})^{\theta}]^{\beta-1} \quad (5)$$

$$\times\left[B([1-e^{-(\lambda x)^2}]^{\theta},\alpha,\beta)\right]^{\gamma-1},$$

by differentiating equation (4), where $B([1-e^{-(\lambda x)^2}]^{\theta},\alpha,\beta)$ is the incomplete beta function and α, β, λ, $\theta > 0$. Thus, the EBBX distribution hazard function can be written as

$$h(x,\alpha,\beta,\gamma,\lambda,\theta)=\frac{2\gamma\lambda^2\theta x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{\theta\alpha-1}}{[B(\alpha,\beta)]^{\gamma}(1-[I_{(1-e^{-(\lambda x)^2})^{\theta}}(\alpha,\beta)]^{\gamma})}$$

$$[1-(1-e^{-(\lambda x)^2})^{\theta}]^{\beta-1}$$

$$\times\left[B([1-e^{-(\lambda x)^2}]^{\theta},\alpha,\beta)\right]^{\gamma-1}. \quad (6)$$

Figures 1 and 2 display the pdf and hazard function of EBBX distribution with various parameter values, respectively. The EBBX distribution fits well in most cases, including bathtub, unimodal, decreasing, and increasing. It is greatly flexible and can be reduced to several sub-models when changing its parameters, as listed in Table 1. For instance, EBBX distribution approaches BBX distribution when $\gamma = 1$ and becomes Burr type X distribution when $\alpha = \beta = \gamma = 1$. Hence, it has been proven that EBBX distribution can cover the characteristics of all the sub-models and is more flexible than the sub-models.

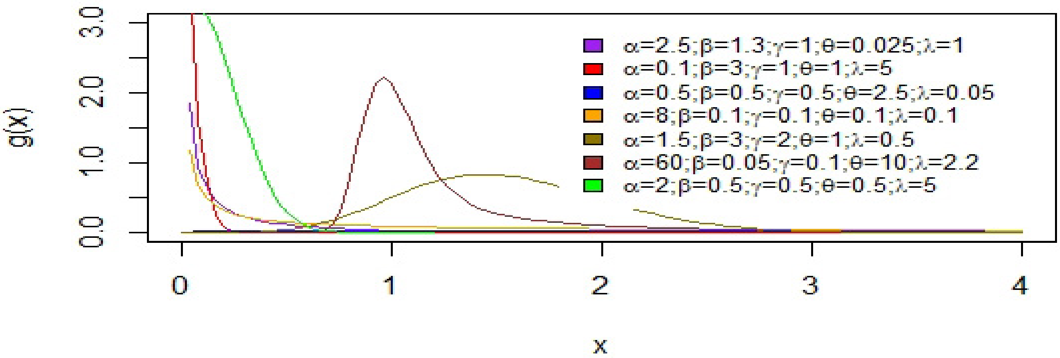

FIGURE 1. EBBX distribution probability density function for different parameters values

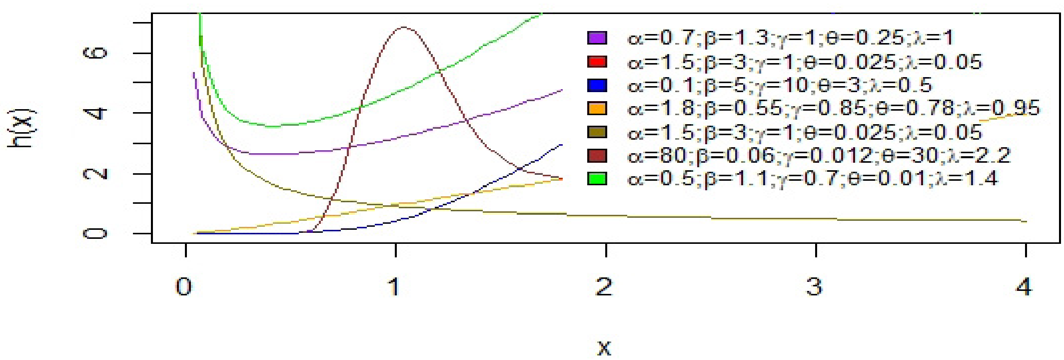

FIGURE 2. EBBX distribution hazard functions for different parameters values

TABLE 1. Exponentiated beta Burr type X distribution sub-models

	Parameters values				
Distribution	α	β	γ	λ	θ
Beta Burr type X			1		
Beta one-parameter Burr type X			1	1	
Burr type X	1	1	1		
Rayleigh	1	1	1		1

MATHEMATICAL PROPERTIES

This section explores several important properties of the EBBX distribution, particularly in its limited behaviour, quantile function, moment, moments generating function (mgf), and order statistics.

When x approaches zero, the limit becomes

$$\lim_{x\to 0}\left(1-e^{-(\lambda x)^2}\right)=0$$

and

$$\lim_{x\to 0}\left[\int_0^{\left(1-e^{-(\lambda x)^2}\right)^\theta} t^{\alpha-1}(1-t)^{\beta-1}dt\right]=0.$$

Thus,

$$\lim_{x\to 0} g(x,\alpha,\beta,\gamma,\lambda,\theta)=\lim_{x\to 0} G(x,\alpha,\beta,\gamma,\lambda,\theta)=0.$$

Meanwhile, as x approaches infinity, we have

$$\lim_{x\to\infty}\left(1-e^{-(\lambda x)^2}\right)=1,$$

since

$$\lim_{x\to\infty} e^{-(\lambda x)^2}=0.$$

Furthermore, we get

$$\lim_{x\to\infty}\left[\int_0^{\left(1-e^{-(\lambda x)^2}\right)^\theta} t^{\alpha-1}(1-t)^{\beta-1}dt\right]=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

$$=B(\alpha,\beta).$$

and we may conclude that

$$\lim_{x\to\infty} g(x,\alpha,\beta,\gamma,\lambda,\theta)=0$$

and

$$\lim_{x\to\infty} G(x,\alpha,\beta,\gamma,\lambda,\theta)=1.$$

For the quantile function of EBBX distribution, we may derive it by inverting its cdf such as

$$Q(u)=x=\frac{1}{\lambda}\left[-\ln\left(1-\left[I^{-1}_{u^{\frac{1}{\gamma}}}(\alpha,\beta)\right]^{\frac{1}{\theta}}\right)\right]^{\frac{1}{2}}, \tag{7}$$

where $u = G(x, \alpha, \beta, \gamma, \lambda, \theta)$ and $I^{-1}_{u^{1/\gamma}}(\alpha,\beta)$ is the inverse function of regularised beta function. Equation (7) is used to simulate EBBX random variable in the Simulation section by letting $\boldsymbol{u}\sim U(0,1)$.

Besides, the r^{th} moment of EBBX distribution can be defined as

$$E(X^r)=\int_0^\infty x^r g(x,\alpha,\beta,\gamma,\lambda,\theta)dx. \tag{8}$$

Using the pdf of the EBBX distribution in equation (5), it becomes

$$E(X^r)=\frac{2\gamma\lambda^2\theta}{[B(\alpha,\beta)]^\gamma}$$
$$\times\int_0^\infty [x^{r+1}e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{\theta\alpha-1}[1-(1-e^{-(\lambda x)^2})^\theta]^{\beta-1} \tag{9}$$
$$\times\left[B\left([1-e^{-(\lambda x)^2}]^\theta,\alpha,\beta\right)\right]^{\gamma-1}\Big]dx,$$

and we may have

$$E(X^r)=\frac{2\gamma\lambda^2\theta}{[B(\alpha,\beta)]^\gamma}$$
$$\times\int_0^\infty\left[x^{r+1}e^{-(\lambda x)^2}\sum_{i=0}^\infty\frac{(-1)^i\Gamma(\beta)}{\Gamma(\beta-i)i!}(1-e^{-(\lambda x)^2})^{\theta(\alpha+i)-1}\right. \tag{10}$$
$$\left.\times\left[B\left([1-e^{-(\lambda x)^2}]^\theta,\alpha,\beta\right)\right]^{\gamma-1}\right]dx.$$

By expanding the binomial term in equation (10), it can then be rewritten as follow,

$$E(X^r) = \frac{2\gamma\lambda^2\theta}{[B(\alpha,\beta)]^\gamma}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{(-1)^{i+j}\Gamma(\beta)\Gamma[\theta(\alpha+i)]}{\Gamma(\beta-i)\Gamma[\theta(\alpha+i)-1]i!\,j!}\right.$$
$$\left.\times\int_0^\infty x^{r+1}\,e^{-(j+1)(\lambda x)^2}\left[B\left([1-e^{-(\lambda x)^2}]^\theta,\alpha,\beta\right)\right]^{\gamma-1}dx\right]. \quad (11)$$

The r^{th} moment of EBBX distribution may be further explored for determining the mean, median, coefficient of variation, kurtosis and skewness of the EBBX distribution.

An alternative way to obtain the r^{th} moment of EBBX distribution is through its moment generating function (mgf). Here, the mgf of the EBBX distribution can be expressed as

$$M_X(t) = \int_0^\infty e^{tx} g(x,\alpha,\beta,\gamma,\lambda,\theta)dx. \quad (12)$$

Using the pdf of the EBBX distribution in equation (5), we obtain

$$M_X(t) = \frac{2\gamma\lambda^2\theta}{[B(\alpha,\beta)]^\gamma}$$
$$\times\int_0^\infty [xe^{-(\lambda x)^2+tx}(1-e^{-(\lambda x)^2})^{\theta\alpha-1}[1-(1-e^{-(\lambda x)^2})^\theta]^{\beta-1} \quad (13)$$
$$\times\left[B\left([1-e^{-(\lambda x)^2}]^\theta,\alpha,\beta\right)\right]^{\gamma-1}]dx.$$

The mgf in equation (13) above then can be further simplified as

$$M_X(t) = \frac{2\gamma\lambda^2\theta}{[B(\alpha,\beta)]^\gamma}$$
$$\times\int_0^\infty\left[xe^{-(\lambda x)^2+tx}\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\beta)}{\Gamma(\beta-i)i!}(1-e^{-(\lambda x)^2})^{\theta(\alpha+i)-1}\right. \quad (14)$$
$$\left.\times\left[B\left([1-e^{-(\lambda x)^2}]^\theta,\alpha,\beta\right)\right]^{\gamma-1}\right]dx,$$

and applying the binomial expansion in equation (14) yields

$$M_X(t) = \frac{2\gamma\lambda^2\theta}{[B(\alpha,\beta)]^\gamma}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{(-1)^{i+j}\Gamma(\beta)\Gamma[\theta(\alpha+i)]}{\Gamma(\beta-i)\Gamma[\theta(\alpha+i)-1]i!\,j!}\right. \quad (15)$$
$$\left.\times\int_0^\infty xe^{-(j+1)(\lambda x)^2+tx}\left[B\left([1-e^{-(\lambda x)^2}]^\theta,\alpha,\beta\right)\right]^{\gamma-1}dx\right].$$

Let $X_1 < X_2 < \ldots < X_n$ be independent random variables from the EBBX distribution, $G(x)$ with its corresponding pdf, $g(x)$. Then, the pdf of the i^{th} order statistic can be defined as

$$g_{X(i)}(x) = \frac{n!}{(i-1)!\,(n-i)!}g(x)[G(x)]^{i-1}[1-G(x)]^{n-i}, \quad (16)$$

Incorporating beta function and binomial expansion, the equation (16) above then becomes

$$g_{X(i)}(x) = \frac{g(x)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[G(x)]^{i+j-1}. \quad (17)$$

By inserting the equations (4) and (5) into equation (17) gives

$$g_{X(i)}(x) = 2\gamma\lambda^2\theta xe^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{\theta\alpha-1}[1-(1-e^{-(\lambda x)^2})^\theta]^{\beta-1}$$
$$\times\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\frac{\left[B\left((1-e^{-(\lambda x)^2})^\theta,\alpha,\beta\right)\right]^{\gamma(i+j)-1}}{B(i,n-i+1)[B(\alpha,\beta)]^{\gamma(i+j)}}.$$

Subsequently, we have

$$g_{X(i)}(x) = 2\gamma\lambda^2\theta xe^{-(\lambda x)^2}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\beta)}{\Gamma(\beta-k)k!}(1-e^{-(\lambda x)^2})^{\theta(\alpha+k)-1}$$
$$\times\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\frac{\left[B\left((1-e^{-(\lambda x)^2})^\theta,\alpha,\beta\right)\right]^{\gamma(i+j)-1}}{B(i,n-i+1)[B(\alpha,\beta)]^{\gamma(i+j)}},$$

by using the binomial expansion. We then simplify the i^{th} order statistic of EBBX distribution as follows:

$$g_{X(i)}(x) = 2\gamma\lambda^2\theta x\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[\binom{n-i}{j}\right.$$
$$\frac{(-1)^{j+k+m}\Gamma(\beta)\Gamma[\theta(\alpha+k)]}{\Gamma(\beta-k)\Gamma[\theta(\alpha+k)-m]k!\,m!}e^{-(m+1)(\lambda x)^2} \quad (18)$$
$$\left.\times\frac{\left[B\left((1-e^{-(\lambda x)^2})^\theta,\alpha,\beta\right)\right]^{\gamma(i+j)-1}}{B(i,n-i+1)[B(\alpha,\beta)]^{\gamma(i+j)}}\right].$$

PARAMETER ESTIMATION

Maximum likelihood approach is the most common approach for estimating the distribution parameters (Phoong & Ismail 2015). Let $x_1, x_2, \ldots, x_n$ be a random sample of EBBX distribution with size n, and then the log-likelihood function can be expressed as

$$l(\alpha,\beta,\gamma,\lambda,\theta) = \sum_{i=0}^{n}[\ln(2)+\ln(\gamma)+2\ln(\lambda)+\ln(\theta)+\ln(x_i)$$
$$-\lambda^2x_i^2] \quad (19)$$
$$+(\theta\alpha-1)\sum_{i=0}\ln(1-e^{-(\lambda x_i)^2})+(\beta-1)\sum_{i=0}\ln[1-(1-e^{-(\lambda x_i)^2})^\theta]$$
$$+(\gamma-1)\sum_{i=0}^{n}\ln\left[\int_0^{(1-e^{-(\lambda x_i)^2})^\theta}t^{\alpha-1}(1-t)^{\beta-1}dt\right].$$

To obtain the maximum likelihood estimation (MLE) of the parameters of EBBX distribution, we differentiate equation (19) partially with respect to each parameter and equate them to zero. We obtain the equations herewith.

$$\frac{\partial l}{\partial \alpha} = \theta \sum_{i=0}^{n} \ln(1 - e^{-(\lambda x_i)^2}) - \gamma n[\psi(\alpha) - \psi(\alpha + \beta)] + \sum_{i=1}^{n} \frac{(\gamma - 1)\left[N - \lim_{\epsilon \to 0} \int_{\epsilon}^{\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}} t^{\alpha-1}(1-t)^{\beta-1} \ln(t)\, dt\right]}{B((1 - e^{-(\lambda x_i)^2})^{\theta}, \alpha, \beta)}, \quad (20)$$

$$\frac{\partial l}{\partial \beta} = -\gamma n[\psi(\beta) - \psi(\alpha + \beta)] + \sum_{i=0}^{n} \ln[1 - (1 - e^{-(\lambda x_i)^2})^{\theta}] + \sum_{i=1}^{n} \frac{(\gamma - 1)\left[N - \lim_{\epsilon \to 0} \int_{\epsilon}^{\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}} t^{\alpha-1}(1-t)^{\beta-1} \ln(1-t)\, dt\right]}{B((1 - e^{-(\lambda x_i)^2})^{\theta}, \alpha, \beta)}, \quad (21)$$

$$\frac{\partial l}{\partial \gamma} = \frac{n}{\gamma} - n\ln[B(\alpha, \beta)] + \sum_{i=0}^{n} \ln\left[B\left((1 - e^{-(\lambda x_i)^2})^{\theta}, \alpha, \beta\right)\right], \quad (22)$$

$$\begin{aligned}\frac{\partial l}{\partial \lambda} &= \frac{2n}{\lambda} - 2\lambda \sum_{i=0}^{n} x_i^2 + 2\lambda(\theta\alpha - 1) \sum_{i=0}^{n} \frac{x_i^2}{e^{(\lambda x_i)^2} - 1} \\ &+ 2\lambda\theta(1 - \beta) \sum_{i=0}^{n} \frac{x_i^2 e^{-(\lambda x_i)^2} \left(1 - e^{-(\lambda x_i)^2}\right)^{\theta-1}}{1 - \left(1 - e^{-(\lambda x_i)^2}\right)^{\theta}} \\ &+ 2\lambda\theta(\gamma - 1) \sum_{i=0}^{n} \frac{x_i^2 e^{-(\lambda x_i)^2} \left(1 - e^{-(\lambda x_i)^2}\right)^{\theta\alpha-1}}{B\left(\left(1 - e^{-(\lambda x_i)^2}\right)^{\theta}, \alpha, \beta\right)} \\ &\cdot \left[1 - \left(1 - e^{-(\lambda x_i)^2}\right)^{\theta}\right]^{\beta-1},\end{aligned} \quad (23)$$

and

$$\begin{aligned}\frac{\partial l}{\partial \theta} &= \frac{n}{\theta} + \alpha \sum_{i=0}^{n} \ln(1 - e^{-(\lambda x_i)^2}) + (1 - \beta) \sum_{i=0}^{n} \frac{\ln(1 - e^{-(\lambda x_i)^2})}{(1 - e^{-(\lambda x_i)^2})^{-\theta} - 1} \\ &+ (\gamma - 1) \sum_{i=0}^{n} \frac{(1 - e^{-(\lambda x_i)^2})^{\theta\alpha} \ln(1 - e^{-(\lambda x_i)^2})}{B((1 - e^{-(\lambda x_i)^2})^{\theta}, \alpha, \beta)} \\ &\left[1 - (1 - e^{-(\lambda x_i)^2})^{\theta}\right]^{\beta-1},\end{aligned} \quad (24)$$

respectively. Due to the complicated system of equations, a numerical method is needed to obtain the MLE of the parameters of EBBX distribution. We have selected the Broyden–Fletcher–Goldfarb–Shanno (BFGS) method as it is often used for unconstrained optimisation of nonlinear functions and is proven to be the most effective method among all quasi-Newton methods (Hery, Ibrahim & June 2014). The R software is chosen for handling this issue.

SIMULATION

To assess the performance of the MLE of the EBBX distribution parameters, we carry out a simulation study along with three sets of parameter values and various sample sizes (n = 50, 150, 300). The three parameter sets are (α, β, γ, λ, θ) = {(80, 0.06, 0.012, 2.2, 30), (0.7, 1.3, 1.0, 1.0, 0.25), (0.1, 5.0, 10, 0.5, 3.0)} for covering the hazard functions in the shapes of unimodal, bathtub and increasing, respectively, as shown in Figure 2. Firstly, we generate the EBBX random variable using its quantile function in equation (7) and then the data are fitted using EBBX distribution. We repeat 2000 times the similar procedure for all cases we consider. The average, root mean square error (RMSE) and bias of the parameter estimations are then recorded in Table 2. Generally, we notice that all the bias values are less than 0.17, while the RMSE values are less than 0.54. The values of bias and RMSE decrease as sample size increases. Besides, for all three sets of parameter values, the average values are close to the true value, RMSE values are decreased toward zero, and bias are close to zero as sample size increase. Hence, we may conclude that the parameter estimators considered are asymptotically unbiased. On the whole, MLE approach is appropriate for estimating EBBX distribution parameters.

APPLICATION

For illustration, we investigate the performance of EBBX distribution through two real data sets. The first data set consists of the failure time of 85 aircraft windshields (Tahir et al. 2015) and the second data set is the failure time reported in '*Weibull Models*' (Murthy, Xie & Jiang 2004). We compare the fitness of EBBX distribution with its sub-models, some extended Burr type X distributions inclusive of the WBX, BBX, EGBX, GBX, and BX, and four nonnested distributions including beta Burr type XII (BBXII) (Paranaiba et al. 2011), exponentiated Burr type XII Poisson (EBXIIP) (Da Silva et al. 2015), exponentiated Weibull Burr type XII (EWBXII) (Abouelmagd, Hamed & Afify 2017), and generalised Marshall-Olkin extended Burr-XII (GMOBXII) (Handique & Chakraborty 2018) distributions.

TABLE 2. Average, RMSE, and bias for different set of parameter values

Set 1									
	α=80			β=0.06			=0.012		
	Average	RMSE	Bias	Average	RMSE	Bias	Average	RMSE	Bias
n=50	80.00002	0.00036	0.00002	0.11980	0.11223	0.05980	0.01553	0.01681	0.00353
n=150	80.00002	0.00010	0.00002	0.09233	0.06213	0.03233	0.01265	0.00267	0.00065
n=300	80.00001	0.00005	0.00001	0.07897	0.03920	0.01897	0.01219	0.00124	0.00019
	λ=2.2			θ=30					
	Average	RMSE	Bias	Average	RMSE	Bias			
n=50	2.23978	0.13422	0.03978	30.00007	0.00095	0.00007			
n=150	2.21051	0.05291	0.01051	30.00005	0.00027	0.00005			
n=300	2.20255	0.02344	0.00255	30.00003	0.00012	0.00003			
Set 2									
	α=0.7			β=1.3			=1.0		
	Average	RMSE	Bias	Average	RMSE	Bias	Average	RMSE	Bias
n=50	0.77322	0.21323	0.07322	1.33763	0.47939	0.03763	1.10527	0.23259	0.10527
n=150	0.75005	0.12484	0.05005	1.33100	0.27552	0.03100	1.05633	0.13767	0.05633
n=300	0.73702	0.08764	0.03702	1.33335	0.18827	0.03335	1.03948	0.09716	0.03948
	λ=1.0			θ=0.25					
	Average	RMSE	Bias	Average	RMSE	Bias			
n=50	1.16194	0.53389	0.16194	0.24316	0.10585	-0.00684			
n=150	1.03535	0.24725	0.03535	0.23731	0.05293	-0.01269			
n=300	1.00535	0.15147	0.00535	0.23966	0.03430	-0.01034			
Set 3									
	α=0.1			β=5.0			=10		
	Average	RMSE	Bias	Average	RMSE	Bias	Average	RMSE	Bias
n=50	0.11211	0.03979	0.01211	5.00070	0.10733	0.00070	10.00009	0.00498	0.00009
n=150	0.10293	0.01492	0.00293	5.00012	0.03336	0.00012	9.99994	0.00122	-0.00006
n=300	0.10217	0.01260	0.00217	4.99824	0.04293	-0.00176	9.99996	0.00190	-0.00004
	λ=0.5			θ=3.0					
	Average	RMSE	Bias	Average	RMSE	Bias			
n=50	0.50513	0.04345	0.00513	2.98549	0.32343	-0.01451			
n=150	0.50176	0.02881	0.00176	2.99898	0.10065	-0.00102			
n=300	0.50098	0.01209	0.00098	2.99201	0.13100	-0.00799			

We use the maximum likelihood approach to estimate the parameters of EBBX distribution. The BFGS method is selected to obtain the parameters' MLE. The initial values are chosen by fitting the model to the Kaplan-meier survival function. It is then followed by the use of chi-square goodness of fit tests, negative log-likelihood (-*l*), Akaike information criteria (AIC) (Akaike 1974), and Bayesian information criteria (BIC) (Schwarz 1978) for assessing the goodness of fit of all competing models. Chi-square goodness of fit test is used to test if the data came from a specific distribution. At the same time, AIC and BIC are used to examine the model fit and model selection. AIC and BIC are the most widely used goodness of fit test statistics in model selection. AIC emphasises the model performance more, and BIC penalises the model complexity more. The smaller value of these criteria indicates a better fit. The chi-square goodness of fit test results is presented in Tables 3 and 5, while the MLEs, negative log-likelihood (-*l*), AIC, and BIC of all competing models are presented in Tables 4 and 6.

Meanwhile, the survival functions are plotted in Figures 3 and 5 and the hazard functions are plotted in Figures 4 and 6. The first data set shows that all competing models can fit the data well as their *p*-values are greater than 0.025. EBBX distribution is the best-fitted model among its sub-models and extended Burr type X distributions as it has the second smallest values in all criteria and the non-nested model EWBXII distribution has the smallest values in all criteria. Besides, it can be considered as an alternative model of EWBXII distribution as the difference between the EBBX distribution and EWBXII distribution for all criteria are small. Meanwhile, for the second data set, all competing models except BBXII distribution can fit the data well as their *p*-values are greater than 0.025 while the *p*-value of BBXII distribution is less than 0.025. Besides, EBBX distribution is comparable with all competing models since there is not much difference in the values of all criteria used. In conclusion, the EBBX distribution fits both data sets well.

TABLE 3. Chi-Square goodness of fit test for first data set

Model	Test statistics	*p-value*
EBBX	6.59134	0.1591
WBX	3.24228	0.6627
BBX	3.78341	0.581
EGBX	4.22105	0.518
GBX	3.78969	0.7051
BX	3.46435	0.839
BBXII	7.19548	0.1259
EBXIIP	2.9798	0.5612
EWBXII	5.60685	0.2305
GMOBXII	3.14854	0.6771

TABLE 4. MLEs, , AIC, BIC, CAIC and HQ for first data set

Model	MLEs	-l	AIC	BIC
EBBX	$\hat{\alpha} = 0.00955$ $\hat{\beta} = 150.49943$ $\hat{\gamma} = 3.13048$ $\hat{\lambda} = 0.29581$ $\hat{\theta} = 38.48913$	125.1093	260.2186	272.3726
WBX	$\hat{\alpha} =109.62902$ $\hat{\beta} = 0.108693$ $\hat{\lambda} = 0.05065$ $\hat{\theta} = 11.144093$	129.3485	266.6970	276.4203
BBX	$\hat{\alpha} =11.76432$ $\hat{\beta} = 0.42007$ $\hat{\lambda} = 0.57382$ $\hat{\theta} = 0.08240$	130.0671	268.1342	277.8574
EGBX	$\hat{\alpha} = 11.19570$ $\hat{\beta} = 0.34823$ $\hat{\lambda} = 0.19342$ $\hat{\theta} = 2.76908$	127.2304	262.4608	272.1841
GBX	$\hat{\gamma} = 0.41840$ $\hat{\lambda} = 0.57497$ $\hat{\theta} = 0.94562$	130.0599	266.1199	273.4123
BX	$\hat{\lambda} = 0.38012$ $\hat{\theta} = 1.19883$	130.4696	264.9391	269.8008
BBXII	$\hat{\alpha} = 6.64048$ $\hat{\beta} = 3.48602$ $\hat{\gamma} = 4.00343$ $\hat{\lambda} = 0.35704$ $\hat{\theta} = 4.93921$	126.7847	263.5693	275.7234
EBXIIP	$\hat{\alpha} = 9.15200$ $\hat{\beta} = 2.74395$ $\hat{\gamma} = 0.55735$ $\hat{\lambda} = 9.71469$ $\hat{\theta} = 4.35239$	129.7183	269.4367	281.5908
EWBXII	$\hat{\alpha} = 6.77928$ $\hat{\beta} = 3.12810$ $\hat{\gamma} = 0.75182$ $\hat{\lambda} = 0.02549$ $\hat{\theta} = 0.14972$	124.3476	258.6951	270.8492
GMOBXII	$\hat{\alpha} = 130.39520$ $\hat{\beta} = 1.90189$ $\hat{\lambda} = 1.66131$ $\hat{\theta} = 4.11653$	128.0537	264.1073	273.8306

TABLE 5. Chi-square goodness of fit test for second data set

Model	Test Statistics	*p-value*
EBBX	7.1297	0.0283
WBX	8.8194	0.0318
BBX	7.1933	0.066
EGBX	6.8325	0.0774
GBX	4.0379	0.4009
BX	3.7776	0.5819
BBXII	7.6158	0.0222
EBXIIP	3.7795	0.1511
EWBXII	6.0054	0.0497
GMOBXII	4.1764	0.2430

TABLE 6. MLEs, , AIC, BIC, CAIC and HQ for second data set

Model	MLEs	*-l*	AIC	BIC
EBBX	$\hat{\alpha}=0.00641$ $\hat{\beta}=127.79513$ $\hat{\gamma}=4.07587$ $\hat{\lambda}=0.166093$ $\hat{\theta}=0.00735$	127.2269	264.4537	276.6670
WBX	$\hat{\alpha}=0.00609$ $\hat{\beta}=3.84535$ $\hat{\lambda}=0.06725$ $\hat{\theta}=0.07293$	128.0732	264.1464	273.9170
BBX	$\hat{\alpha}=0.02737$ $\hat{\beta}=125.32588$ $\hat{\lambda}=0.30250$ $\hat{\theta}=37.09216$	127.2328	262.4657	272.2363
EGBX	$\hat{\alpha}=10.43517$ $\hat{\beta}=0.34956$ $\hat{\lambda}=0.19754$ $\hat{\theta}=2.78325$	129.5429	267.0857	276.8563
GBX	$\hat{\gamma}=0.40096$ $\hat{\lambda}=0.58455$ $\hat{\theta}=0.93547$	132.0494	270.0989	277.4268

BX	$\hat{\lambda} = 0.378329$ $\hat{\theta} = 1.19405$	132.4766	268.9532	273.8385
BBXII	$\hat{\alpha} = 4.82107$ $\hat{\beta} = 3.89308$ $\hat{\gamma} = 1.02918$ $\hat{\lambda} = 0.35632$ $\hat{\theta} = 5.05567$	129.6880	269.3761	281.5893
EBXIIP	$\hat{\alpha} = 11.61529$ $\hat{\beta} = 5.73349$ $\hat{\gamma} = 0.56406$ $\hat{\lambda} = 10.00540$ $\hat{\theta} = 4.27022$	131.6705	273.3409	285.5542
EWBXII	$\hat{\alpha} = 7.23979$ $\hat{\beta} = 3.13659$ $\hat{\gamma} = 0.84318$ $\hat{\lambda} = 0.03761$ $\hat{\theta} = 0.12841$	126.5972	263.1944	275.4076
GMOBXII	$\hat{\alpha} = 182.39431$ $\hat{\beta} = 2.13133$ $\hat{\lambda} = 1.51739$ $\hat{\theta} = 4.67072$	129.6934	267.3868	277.1574

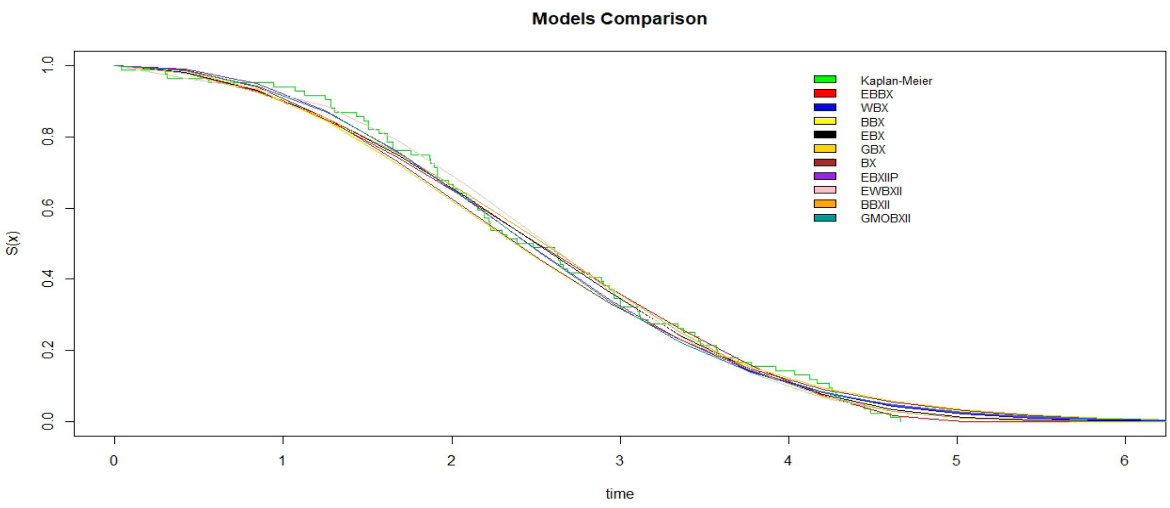

FIGURE 3. Estimated survival function for first data set

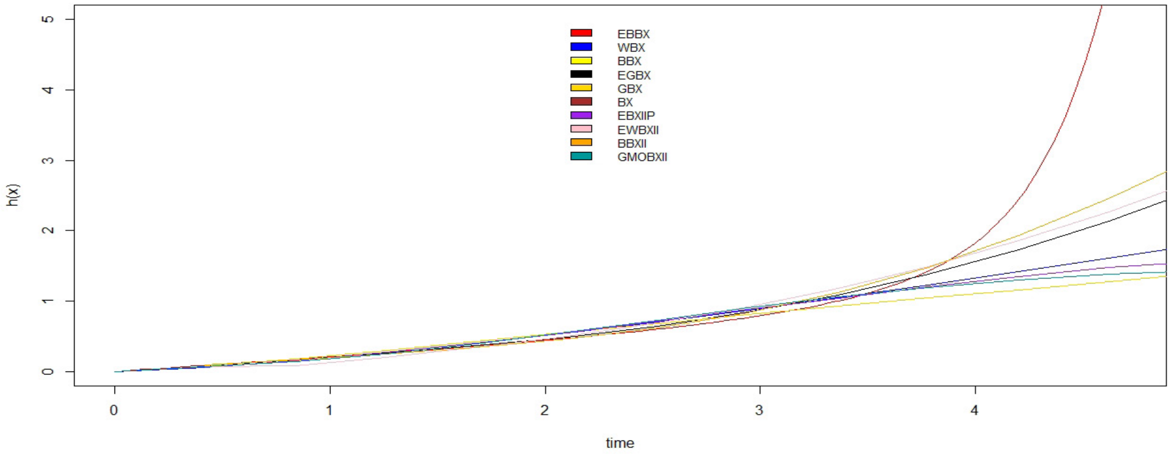

FIGURE 4. Estimated hazard function for first data set

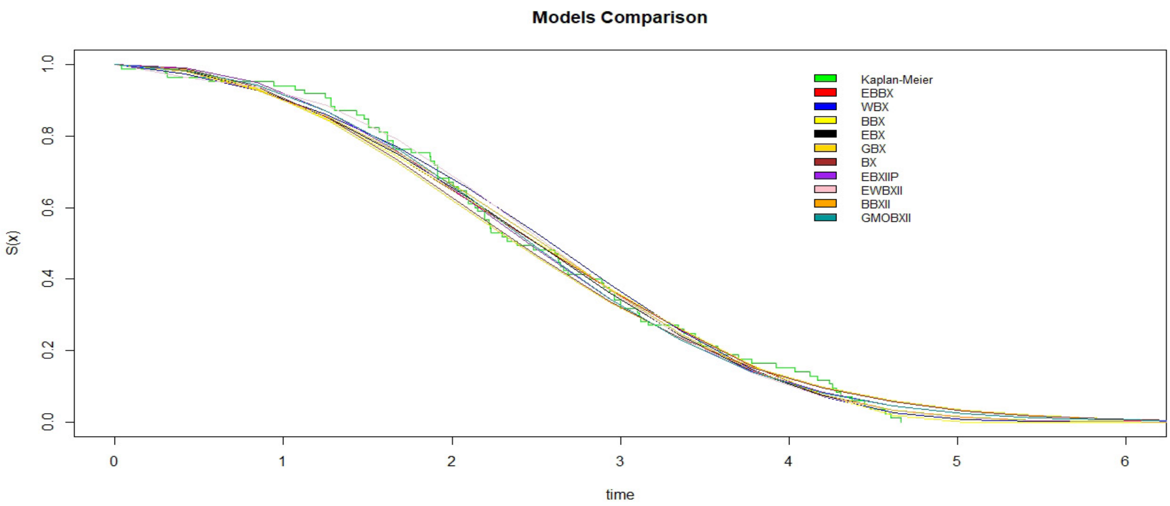

FIGURE 5. Estimated survival function for second data set

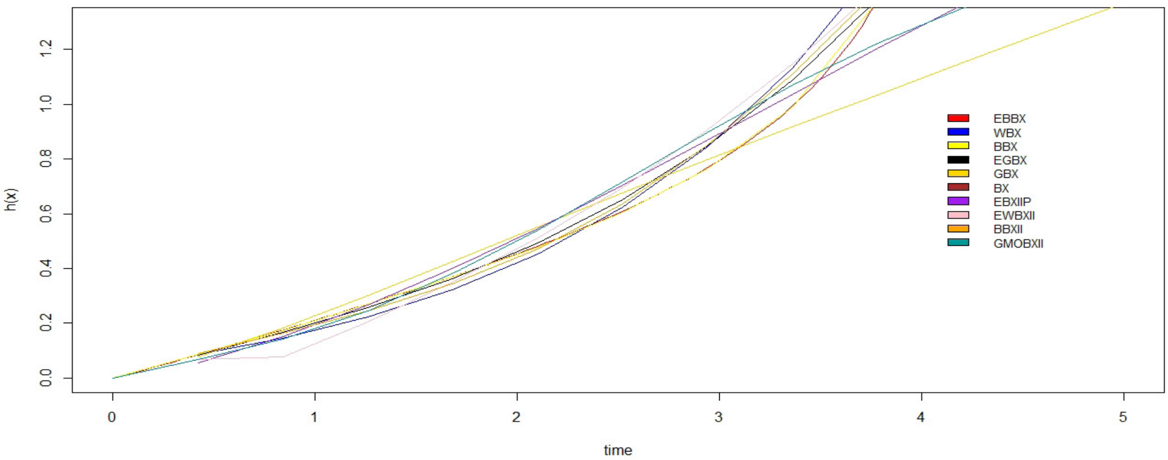

FIGURE 6. Estimated hazard function for second data set

Conclusion

This study aims to introduce a new continuous distribution, which provides greater flexibility, to overcome the deficiency of the existing extended Burr type X distributions. That is, these extended Burr type X distributions can only model the hazard function in shapes of decreasing, increasing and bathtub, but not for unimodal. Hence, we propose a new five-parameter distribution, namely exponentiated beta Burr type X distribution which extends the beta Burr type X distribution. We obtain its pdf, cdf, and hazard function. The hazard function has various forms, such as increasing, decreasing, bathtub, and unimodal. We explore the mathematical properties of EBBX distribution, which includes quantile function, moment, moment generating function, and order statistics. We use the maximum likelihood method to estimate the parameters of EBBX distribution. A simulation study with varying sample sizes and parameters is conducted to examine the performance of EBBX distribution. The parameter values chosen covered unimodal, bathtub, and increasing hazard functions. The result shows that EBBX performed well in modelling hazard functions in various shapes and the maximum likelihood approach can be used to estimate the parameters of EBBX distributions. The real data sets illustrate that the suggested distribution is comparable to all competing distributions. In conclusion, the proposed EBBX distribution provides a better fit in modelling lifetime data and can be used as an alternative model for all the competing models, especially the four nonnested models. It may model all shapes of hazard functions, including the unimodal. In addition, the proposed distribution can be applied to model lifetime data in different fields, such as engineering and the medical fields. However, the paper did not consider the presence of censored observations and covariates. Thus, future research should consider the presence of censored observations and covariates.

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