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On the Impact of Asymmetric Dependence in the Actuarial Pricing of Joint Life Insurance Policies

(Kesan Kebersandaran Asimetri dalam Harga Aktuari Polisi Insurans Hayat Tercantum)

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ABSTRACT

Multipopulation mortality modeling is a significant research problem in actuarial science. Mortality functions involving multiple lives are also essential to determine the pricing of premiums. Moreover, the lifetime models based on dependence and asymmetry are more realistic. Hence, this paper applies an asymmetric copula model, Generalized FGM (GFGM) to model the bivariate joint distribution of future lifetimes. Premiums of first-death life insurance products are calculated based on the proposed model and compared with independent and symmetrical models. The results display that asymmetry has a significant effect on premium calculations. Also, it is concluded that the lowest premiums are generally in asymmetric lifetime models. This paper also provides analytical examples for the proposed model with Gompertz's marginal law.

Keywords: Asymmetric dependence; copula; insurance; joint life (first-death); premium

ABSTRAK

Pemodelan mortaliti populasi berbilang merupakan permasalahan penyelidikan yang penting dalam bidang sains aktuari. Fungsi mortaliti yang melibatkan model hayat berbilang juga berperanan untuk menentukan harga premium. Selain itu, model masa-hayat berdasarkan kebersandaran dan asimetri adalah lebih realistik. Oleh itu, makalah ini menggunakan model kopula asimetri dan *Generalized* FGM (GFGM) untuk memodelkan taburan tercantum bivariat bagi jangka hayat masa hadapan. Premium bagi produk insurans hayat kematian-pertama dihitung berdasarkan model yang dicadangkan dan dibandingkan dengan model tak bersandar dan simetri. Keputusan menunjukkan bahawa asimetri mempunyai kesan yang signifikan ke atas pengiraan premium. Selain itu, dapat disimpulkan bahawa premium terendah kebiasaannya ditunjukkan dalam model masa hayat asimetri. Kajian ini juga menyediakan contoh analisis bagi model yang dicadangkan menggunakan marginal Gompertz.

Kata kunci: Hayat tercantum (kematian-pertama); insurans; kebersandaran asimetri; kopula; premium

INTRODUCTION

There has been increased research on dependence modeling by copulas in recent years. A great deal of literature has emerged in the application areas of copulas, e.g., in insurance (Carriere 2000; Denuit & Cornet 1999; Frees, Carriere & Valdez 2006; Hsieh, Tsai & Wang 2020; Lee, Lee & Kim 2014; Luciano, Spreeuw & Vigna 2008; Shemyakin & Youn 2006), economics and finance (Jung, Kim & Kim 2008; Uhm, Kim & Jung 2012). The main reason for the interest in copulas is that copulas allow the dependence of the joint distribution to be modeled independently of the marginal distributions without the assumption of normality. For example, in the actuarial field, couples' future lifetimes can depend on exposure to joint disaster and broken-heart syndrome (Jagger & Sutton 1991). For this purpose, the copula is a popular statistical modeling tool used to model the joint behavior of correlated random variables.

Sklar Theorem (1959) defines copula in the literature. This theorem states that the joint distribution functions with continuous marginals $F(x)$ and $G(y)$ can be uniquely written via a copula C:$[0,1]^2 \rightarrow 0,1]$ in the form $H(x, y) = C(F(x), G(y))$, or $C(u,v) = H(F^{(-1)}(\mathrm{u}), G^{(-1)}(v))$ for uniform marginals. It satisfies the distribution function properties. For the details, it can be looked to Nelsen (2007).

On the other hand, there are concepts of symmetry and asymmetry in dependency. A copula is said to be asymmetric if $C(u, v) \neq C(v, u)$ for all (u,v) in $[0, 1]^2$, if the distribution exhibits different behaviors in the upper left and the lower right triangle of the unit square. Suppose an example is given from the actuarial field. In that case, symmetrical dependence means that for the example of dependent lifetime data of married couples, the first death of the spouses is male and the female has the same effect on the survivor's lifetime. However, this effect is not the same, which shows the dependence among couples' lifetime is asymmetrical. In other words, asymmetric dependence on lifetime data may occur since men are affected more or less than women.

In the literature, it is noteworthy to mention that despite extensive studies on symmetry by subject dependence, very few studies on asymmetry exist. There are some studies on asymmetric dependence in applications such as hydrology and finance (Ang, Chen & Xing 2006; Bücher, Irresberger & Weiss 2017; Harvey & Siddique 2000). However, it is interesting that asymmetric dependence is rarely studied in actuarial and insurance fields despite its usefulness. Zhu, Tan and Wang (2017) computed the swap premiums under asymmetric Hierarchical Archimedean Copula (HAC). Dufresne et al. (2018) and Lu (2017) model the lifetime data of married couples with the Archimedean copulas. However, they use symmetric Copulas, although they state that the data is asymmetric. In line with the literature, the previous study by Kara (2021) proposes showing the effect of asymmetric dependence on actuarial insurance premiums under joint last survivor status for the whole life insurance policies. The results show that premiums are influenced by asymmetric dependency. The symmetrical and asymmetric lifetimes are assumed to be modeled by the FGM and Type II GFGM copula families, respectively.

In the current study, to see the effect of asymmetric dependence in premium calculations, the previous study has been extended for other insurance products (whole life, term life, and endowment life insurance policies) under the first death status. FGM and Type II GFGM copula families are used for symmetric and asymmetric lifetime models since they are theoretically simple and represent dependent structures with little correlation. Firstly, the joint survival and mortality functions are derived for these models based on Gompertz's law using the copula. Later, net single premium calculations for selected ages and parameter values are performed according to the whole, term, and endowment life insurances. The results show significant differences in the premiums calculated under symmetric and asymmetric dependent lifetime. Also, it is concluded that as the correlation increases, the premium coefficients decrease. It can be said that while asymmetry is visibly effective in whole and term life insurance policies, it has a more negligible effect in endowment insurances. Finally, the compared results demonstrate that the studied dependence models are sensitive to the various Spearman values and age differences.

The remainder of this paper is structured as follows: In the next section, we introduce the symmetric and asymmetric copula models. Subsequent section presents the actuarial methodology in which we give the definitions of the net single premium for various life insurance products. In the section that follows is the original part of this paper. Firstly, we derive the joint survival and mortality functions for the studied symmetric and asymmetric copula models. Then, using these inferences, results for the net single premiums of various life insurance products are illustrated. Last section concludes the article. All proofs are deferred to an appendix.

SYMMETRIC AND ASYMMETRIC COPULA MODELS

One of the most widely studied parametric copula families is the Farlie-Gumbel-Morgenstern (FGM) copula family and the distribution function is defined as follows (Nelsen 2007):

$$C_\theta^{FGM} = uv + \theta uv(1-u)(1-v) \tag{1}$$

where $\theta \in [-1,1]$ is dependence parameter. The FGM distribution does not allow to model the high dependence since Spearman's ρ correlation coefficient is in interval [-1/3,1/3]. To overcome this problem, Generalized FGM (GFGM) copula families have been developed by some authors (Rodriguez-Lallena & Úbeda-Flores 2004; Shubina & Lee 2004). For example, Bairamov and Kotz (2002), Bairamov, Kotz and Bekci (2001), and Huang and Kotz (1999), also studied the special cases of the GFGM copula family. For some selected parameter values, they obtained ρ values greater than 0.33. These results on the GFGM function are briefly summarized by Güven and Kotz (2008).

In the following, the bivariate GFGM family introduced by Rodriguez-Lallena and Úbeda-Flores (2004) is defined for $0 \leq u,\ v \leq 1$ as follows:

$$C(u,v) = uv + f(u)g(v) \quad (2)$$

Here, the general form called Type II has been obtained as below by choosing $f(u) = \sqrt{\theta}u^b(1-u)^\alpha$ and $g(v) = \sqrt{\theta}v^b(1-v)^\beta$.

$$C(u,v) = uv + \theta u^a v^b (1-u)^\alpha (1-v)^\beta \quad (3)$$

where $\alpha \geq 1$ and $\beta \geq 1$ are asymmetry parameters; a and b are any given values. θ is the dependency parameter that contains interval [-1,1] for all $a, b, \alpha, \beta \geq 1$. Moreover, the FGM copula family is obtained for $a = b = c = d = 1$. θ is in the interval $[-\frac{1}{\max\{\tau\gamma,\varphi\delta\}}, -\frac{1}{\min\{\tau\delta,\varphi\gamma\}}]$ for the Type II-GFGM. Here the τ, φ, γ, and δ equations restate for $a = b$. On the other hand, the Spearman's ρ correlation is given by

$$\rho_C = 12\theta Beta(b+1,\alpha+1)Beta(b+1,\beta+1) \quad (4)$$

where $Beta(.\,,.)$ is the Beta function defined by $Beta(k, l) = \int_0^1 x^{k-1}(1-x)^{l-1}dx,\ k,l > 0$.

Jung, Kim and Kim (2008) and Uhm, Kim & Jung (2012) show that Type II is an asymmetrical copula function. This paper studies the Type II-GFGM copula to evaluate premium calculations in the next section.

ACTUARIAL BACKGROUND

Bowers et al. (1997) and Gerber (1997) define the actuarial notations for T_x, the future lifetime random variable of an individual aged x. The distribution function and survival function are denoted by $F_X(t) = P(T_x \leq t)$ and $S_X(t) = P(T_x > t)$, respectively. In actuarial literature, these functions are known as death and survival probabilities ${}_tq_x$ and ${}_tp_x$, respectively.

In a single-life state, the relations between mortality and survival functions are ${}_tp_x = e^{-\int_0^t \mu(x+s)ds}$, and ${}_t\mu_x = -\frac{\partial}{\partial t}\ln[{}_tp_x]$ for $0 \leq x < w$ and $0 \leq t < w - x$. Here w is the last age.

Similar notations are given for the joint-life state by Dickson et al. (2013) and Menge and Glover (1938). The lifetimes of two individuals ages x and y are indicated by T_x and T_y, respectively, the joint lifetime is denoted by $T_{xy} = \min\{T_x, T_y\}$. The joint cumulative distribution function and survival function of T_{xy} are defined as $F_{xy}(t) = P(T_{xy} \leq t)$ and $S_{xy}(t) = P(T_{xy} > t)$, respectively. For T_{xy} random variable, the joint probability density, and the joint mortality functions are given by $f_{xy}(t) = \frac{\partial}{\partial t}F_{xy}(t)$ and ${}_t\mu_{xy} = f_{xy}(t)/S_{xy}(t)$. The relations between mortality and survival functions are defined as following by actuarial notations

$$ {}_tp_{xy} = e^{-\int_0^t \mu_{x+s:y+s}ds} \quad (5)$$

$$ {}_t\mu_{xy} = -\frac{\partial}{\partial t}\ln[{}_tp_{xy}] \quad (6)$$

THE SURVIVAL FUNCTION FOR DEPENDENCE LIFETIME MODELS

In the case of the dependence lifetime, Nelsen (2007) defines the bivariate survival function as $S_{xy}(s,t) = S_x(s) + S_y(t) - 1 + C(1-S_x(s), 1-S_y(t))$. Here, if the joint distribution function is denoted by $H_{xy}(s,t) = P(T_x < s, T_y < t)$ then it can be defined with a C copula function as $H_{xy}(s,t) = C(F_x(s), F_y(t))$ with the marginal functions, $F_x(s) = 1-S_x(s)$, and $F_y(t) = 1-S_y(t)$. Also, by actuarial notations, it can be shown in the form of $S_{xy}(s,t) = p_{x:s,\,y:t}$. In the next section, we evaluate some inferences on premiums for $(x \neq y, s = t)$. Therefore, $p_{x:s,\,y:t}$ can also be expressed based on copula by ${}_tp_{xy} = {}_tp_x + {}_tp_y - 1 + C({}_tq_x, {}_tq_y)$. Here, ${}_sp_x = S_x(s)$, ${}_tp_y = S_y(t)$, ${}_sq_x = 1 - {}_sp_x$ and ${}_tq_y = 1 - {}_tp_y$.

INSURANCE POLICIES AND ACTUARIAL NET PREMIUM CALCULATIONS

We introduce three types of insurance policies in the literature. One of them is a whole life insurance policy that provides one unit of death coverage paid to the policyholder at the end of the death year. The other policy types are term and endowment insurances. In the endowment insurance, while the death coverage is paid, the insured dies during the contract period, i.e., before n or lives in the last at time n; in the term insurance policies, it is paid only if the insured dies before n.

On the other hand, policy types are issued according to the single-life and joint-life situations. The policies in a single life state are issued for only a person. The policies for joint life state can be issued according to the first death (joint life) and the last survivor situations. For the joint-life policies, the death coverage is paid at the end of the year of death in which the first death occurs, while for the last survivor policies, it is paid when the last survivor dies.

This study makes premium calculations for the joint-life (first-death) insurance policies in the next section. The premium calculations have been studied for the last survivor policies in Kara (2021).

Now, actuarial notations related to insurance policies issued for an individual age x are summarized as follows. For life insurance policies, the net single premium is called the Actuarial Present Value (APV) in the literature. It can be seen in Dickson et al. (2013) and Menge and Glover (1938) for further details.

In the single life status, the APVs are defined by $\overline{A}_x = \int_0^\infty v^n{}_tp_x\ \mu_x(t)dt$, $\overline{A}^{\,1}_{x:n\rceil} = \int_0^n v^n{}_tp_x\ \mu_x(t)dt$, and $\overline{A}_{x:n\rceil} = \overline{A}^{\,1}_{x:n\rceil} + v^n{}_tp_x$ for whole, term, and endowment life insurance policies, respectively. The APVs for whole and endowment insurance policies can also be computed by the equations $\overline{A}_x = 1 - \delta\overline{a}_x$, and $\overline{A}_{x:n\rceil} = 1 - \delta\overline{a}_{x:n\rceil}$, respectively. Here $\overline{a}_x$ shows the present value of whole life annuity and it is defined as $\overline{a}_x = \int_0^\infty v^n{}_tp_x dt$.

In the joint-life status, consisting of two individuals ages x and y, the APVs are defined by $\overline{A}_{xy} = \int_0^\infty v^n{}_tp_{xy}\ \mu_{x+t:y+t}dt$, $\overline{A}^{\,1}_{xy:n\rceil} = \int_0^n v^n{}_tp_{xy}\ \mu_{x+t:y+t}dt$ and $\overline{A}_{xy:n\rceil} = \overline{A}^{\,1}_{xy:n\rceil} + v^n{}_tp_{xy}$ for whole, term, and endowment life insurance policies, respectively. Also, the APVs for whole and endowment insurance policies can be computed by the equations $\overline{A}_{xy} = 1 - \delta\overline{a}_{xy}$, and $\overline{A}_{xy:n\rceil} = 1 - \delta\overline{a}_{xy:n\rceil}$, respectively. Here $\overline{a}_{xy}$ shows the present value of whole life annuity with the joint-life and it is defined as $\overline{a}_{xy} = \int_0^\infty e^{-\delta t}{}_tp_{xy}dt$. Here $v^n = e^{-\delta t}$ and the constant δ is force of interest rate.

THE INFERENCES WITH SYMMETRIC AND ASYMMETRIC DEPENDENCE

This section aims to derive the joint survival and mortality functions under the dependence lifetime models by using Equations (5) and (6). The joint survival functions are given for FGM and GFGM models, respectively.

$$ {}_tp_{xy} = {}_tp_x + {}_tp_y - 1 + FGM[1 - {}_tp_x, 1 - {}_tp_y, \theta] \quad (7)$$

$$ {}_tp_{xy} = {}_tp_x + {}_tp_y - 1 + GFGM[1 - {}_tp_x, 1 - {}_tp_y,\ \theta, b, \alpha, \beta] \quad (8)$$

The details of these implications for Gompertz's laws are given in Appendix A. On the other hand, the inferences for the *FGM*(θ) and *GFGM*(θ, b, α, β) copula models constitute the original part of this paper. For this purpose, survival and mortality functions for the first death state are simplified with actuarial notations in the lemmas below. Next, explicit numerical solutions for some parameters are obtained with Mathematica 12.

Lemma 4.1 The survival and mortality functions for the symmetric dependence (*FGM* copula) are given by

$$ {}_tp_{xy} = (1 + \theta(1 - {}_tp_x - {}_tp_y + {}_tp_{x\,t}p_y))$$

$$ {}_t\mu_{xy} = \frac{1}{K}\left({}_t\mu_x\left(K - \frac{{}_tq_y}{{}_tp_y}\theta\right) + {}_t\mu_y\left(K - \frac{{}_tq_x}{{}_tp_x}\theta\right)\right)$$

where $K = \frac{1}{{}_tp_{x\,t}p_y}\left(1 + \theta(1 - {}_tp_x - {}_tp_y + {}_tp_{x\,t}p_y)\right)$.

Lemma 4.2 The survival and mortality functions for the asymmetric dependence (*GFGM* copula) are given by

$$ {}_tp_{xy} = {}_tp_{x\,t}p_y + \theta({}_tp_x)^\alpha({}_tp_y)^\beta(1 - {}_tp_x)^b(1 - {}_tp_y)^b$$

$$ {}_t\mu_{xy} = \left(\frac{1}{{}_tp_{x\,t}p_y + G}\right)\left({}_tp_{x\,t}p_y({}_t\mu_x - {}_t\mu_y) - bG\left(\left(\frac{{}_tp_x}{{}_tq_x}\right)_t\mu_x - \left(\frac{{}_tp_y}{{}_tq_y}\right)_t\mu_y\right) - G(\alpha_t\mu_x + \beta_t\mu_y)\right)$$

where $G = \theta({}_tp_x)^\alpha({}_tp_y)^\beta(1 - {}_tp_x)^b(1 - {}_tp_y)^b$.

THE NET SINGLE PREMIUMS UNDER THE DEPENDENCE LIFETIME

This section aims to explain the effect of asymmetry on actuarial premiums. Throughout the paper, Gompertz (B=0.001, c=1.0887) is used for mortality function. The dependence parameters of *FGM*(θ) and *GFGM*(θ, b = 1.5, α = 2.5, β = 1.8) are derived for the Spearman's correlation values, $\rho \in \{0.1, 0.2, 0.3\}$ using Equation (4). For the dependency models, the selected parameter values and asymmetric measure values are given in Table 1. The graphs, ${}_tp_{xy}$ and ${}_t\mu_{xy}$ for *GFGM* are presented in Figure 1 by ρ = 0.3. using Lemma 4.2.

Next, we examine the pricing of the proposed models by considering the different life insurance policies. Table 2 presents the net single premiums under Spearman's correlations $\rho \in \{0.1, 0.2, 0.3\}$ for Gompertz's laws. The premium plots for ρ = 0.3 are shown in Figure 2. Here the force of interest is taken as 0.06; the period is assumed to be ten years for a term insurance policy.

TABLE 1. The selected parameter values and the asymmetric measures for FGM and GFGM copulas

	Spearman's correlations	Dependence parameters	Asymmetric measures		
	ρ	θ	$\rho_{uv}^{(2)}$	$\rho_{vu}^{(2)}$	$\left\|\rho_{uv}^{(2)} - \rho_{vu}^{(2)}\right\|$
FGM -0.3 ≤ ρ ≤ 0.3 -1 ≤ θ ≤1	0.1	0.3	0.005	0.005	0.0000
	0.2	0.6	0.020	0.020	0.0000
	0.3	0.9	0.045	0.045	0.0000
	ρ	θ	$\rho_{uv}^{(2)}$	$\rho_{vu}^{(2)}$	$\left\|\rho_{uv}^{(2)} - \rho_{vu}^{(2)}\right\|$
GFGM (b = 1.5, α = 2.5, β = 1.8) -0.25177 ≤ ρ ≤ 0.30795 -9.73732 ≤ θ ≤ 11.90980	0.1	3.86744	0.01463	0.00944	0.00519
	0.2	7.73489	0.05855	0.03777	0.02078
	0.3	11.6023	0.13175	0.08499	0.04676

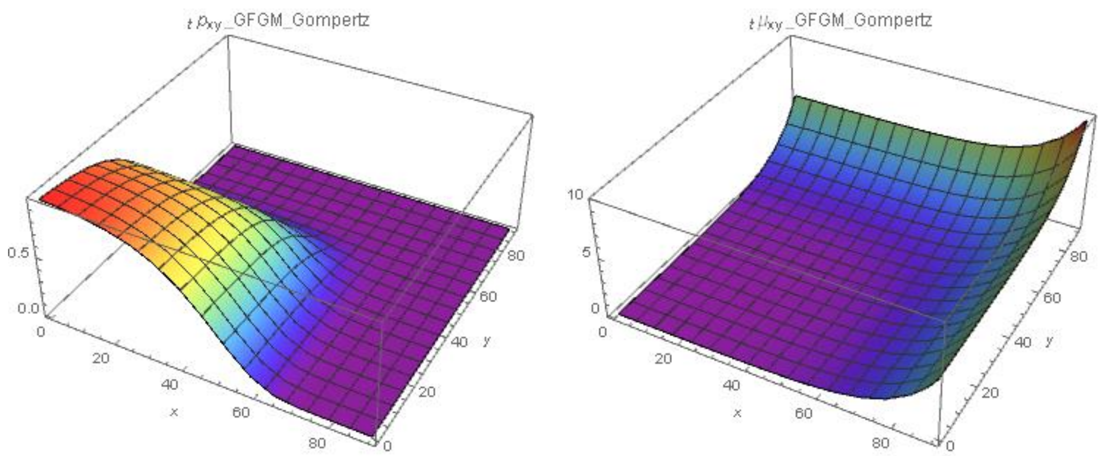

FIGURE 1. The graphs (left panel) p_{xy} and (right panel) $_t\mu_{xy}$ for GFGM model (ρ = 0.3 and t =10)

We draw the following observations based on Table 2. 1) Dependency significantly affects premiums. Indeed, the premiums reduced under symmetric dependence generally decrease even more under asymmetric dependence. 2) Asymmetry on the premiums is more significant in highly correlated lifetimes. It is remarkable that as the correlation value increases, the asymmetry also increases. For example, $\overline{A}_{55:50}$ = 0.78822 and $\overline{A}_{50:55}$ = 0.78934 for ρ = 0.1 , $\overline{A}_{50:55}$ = 0.77447 and $\overline{A}_{55:50}$ = 0.77784 for ρ = 0.3 and, that is, the differences are 0.00112 and 0.00337 by assuming ρ = 0.1 and ρ = 0.3, respectively for GFGM model. 3). The premiums are sensitive to the age differences. Indeed, the symmetrical or asymmetrical effect is more pronounced in increasing age differences. For example, $\overline{A}_{50:55}$ = 0.77447 , $\overline{A}_{50:60}$ = 0.81378 and $\overline{A}_{50:65}$ = 0.85380 by assuming ρ = 0.3 and GFGM model.

TABLE 2. The net single premiums for the joint-life insurance policies

Gompertz		Whole Life ($\overline{A}_{xy}$)											
		Independent				FGM				GFGM			
Female	Male	50	55	60	65	50	55	60	65	50	55	60	65
	$\rho = 0.1$					0.75576	0.78932	0.82563	0.86168	0.75486	0.78822	0.82477	0.86127
50	$\rho = 0.2$	0.76217	0.79509	0.83026	0.86500	0.74934	0.78355	0.82100	0.85837	0.74755	0.78134	0.81927	0.85753
	$\rho = 0.3$					0.74292	0.77778	0.81637	0.85505	0.74024	0.77447	0.81378	0.85380
	$\rho = 0.1$						0.81398	0.84225	0.87194	0.78934	0.81345	0.84152	0.87139
55	$\rho = 0.2$	0.79509	0.81961	0.84716	0.87575		0.80834	0.83733	0.86814	0.78359	0.80728	0.83588	0.86702
	$\rho = 0.3$						0.80270	0.83242	0.86433	0.77784	0.80111	0.83024	0.86266
	$\rho = 0.1$							0.86251	0.88521	0.82640	0.84248	0.86227	0.88479
60	$\rho = 0.2$	0.83026	0.84716	0.86719	0.88918			0.85782	0.88124	0.82253	0.83779	0.85736	0.88040
	$\rho = 0.3$							0.85314	0.87728	0.81867	0.83311	0.85244	0.87728
	$\rho = 0.1$								0.90111	0.86276	0.87272	0.88555	0.90108
65	$\rho = 0.2$	0.86500	0.84716	0.88918	0.90481				0.89742	0.86052	0.86969	0.88193	0.89734
	$\rho = 0.3$								0.89372	0.85828	0.86666	0.87831	0.89361

		Term Life ($\overline{A}^{\;1}_{xy:\overline{n}|}$)											
		Independent				FGM				GFGM			
Female	Male	50	55	60	65	50	55	60	65	50	55	60	65
	$\rho = 0.1$					0.69674	0.75466	0.81080	0.85778	0.69755	0.75592	0.81164	0.85794
50	$\rho = 0.2$	0.70985	**0.76519**	0.81772	0.86173	0.68363	0.74412	0.80388	0.85382	0.68524	0.74664	0.80555	0.85415
	$\rho = 0.3$					0.67052	**0.73359**	0.79695	0.84987	0.67294	**0.73736**	0.79947	0.85036
	$\rho = 0.1$						0.79368	0.83359	0.86967	0.75784	0.79572	0.83427	0.86954
55	$\rho = 0.2$	**0.76519**	**0.80265**	0.84010	0.87392		0.78471	0.82709	0.86543	0.75048	0.78879	0.82844	0.86515
	$\rho = 0.3$						**0.77574**	0.82059	0.86118	**0.74312**	**0.78186**	0.82262	0.86077
	$\rho = 0.1$							0.85883	0.88425	0.81365	0.83531	0.85932	0.88403
60	$\rho = 0.2$	0.81772	0.84010	0.86426	0.88842			0.85340	0.88008	0.80388	0.83052	0.85438	0.87964
	$\rho = 0.3$							0.84797	0.87591	0.80550	0.82574	0.84944	0.87525
	$\rho = 0.1$								0.90086	0.85949	0.87089	0.88480	0.90088
65	$\rho = 0.2$	0.86173	0.87392	0.88842	0.90462				0.89711	0.85724	0.86785	0.88118	0.89715
	$\rho = 0.3$								0.89336	0.85499	0.86482	0.87755	0.89341

		Endowment Life ($\overline{A}_{xy:\overline{n}|}$)											
		Independent				FGM				GFGM			
Female	Male	50	55	60	65	50	55	60	65	50	55	60	65
	$\rho = 0.1$					0.76516	0.79378	0.82710	0.86197	0.76338	0.79213	0.82602	0.86151
50	$\rho = 0.2$	0.77019	**0.79882**	0.83148	0.86524	0.76014	0.78875	0.82273	0.85871	0.75658	0.78544	0.82056	0.85777
	$\rho = 0.3$					0.75512	**0.78371**	0.81836	0.85544	0.74977	**0.77875**	0.81510	0.85404
	$\rho = 0.1$						0.81616	0.84299	0.87210	0.79318	0.81528	0.84213	0.87151
55	$\rho = 0.2$	**0.79882**	**0.82140**	0.84776	0.87587		0.81093	0.83823	0.86832	0.78755	0.80915	0.83650	0.86714
	$\rho = 0.3$						**0.80570**	0.83346	0.86454	**0.78191**	**0.80303**	0.83087	0.86278
	$\rho = 0.1$							0.86277	0.88527	0.82763	0.84308	0.86248	0.88483
60	$\rho = 0.2$	0.83148	0.84776	0.86740	0.88922			0.85814	0.88131	0.82377	0.83840	0.85757	0.88045
	$\rho = 0.3$							0.85352	0.87736	0.81992	0.83372	0.85265	0.87606
	$\rho = 0.1$								0.90113	0.86300	0.87284	0.88560	0.90108
65	$\rho = 0.2$	0.86524	0.87587	0.88922	0.90482				0.89743	0.86076	0.86981	0.88198	0.89735
	$\rho = 0.3$								0.89374	0.85852	0.86678	0.87835	0.89362

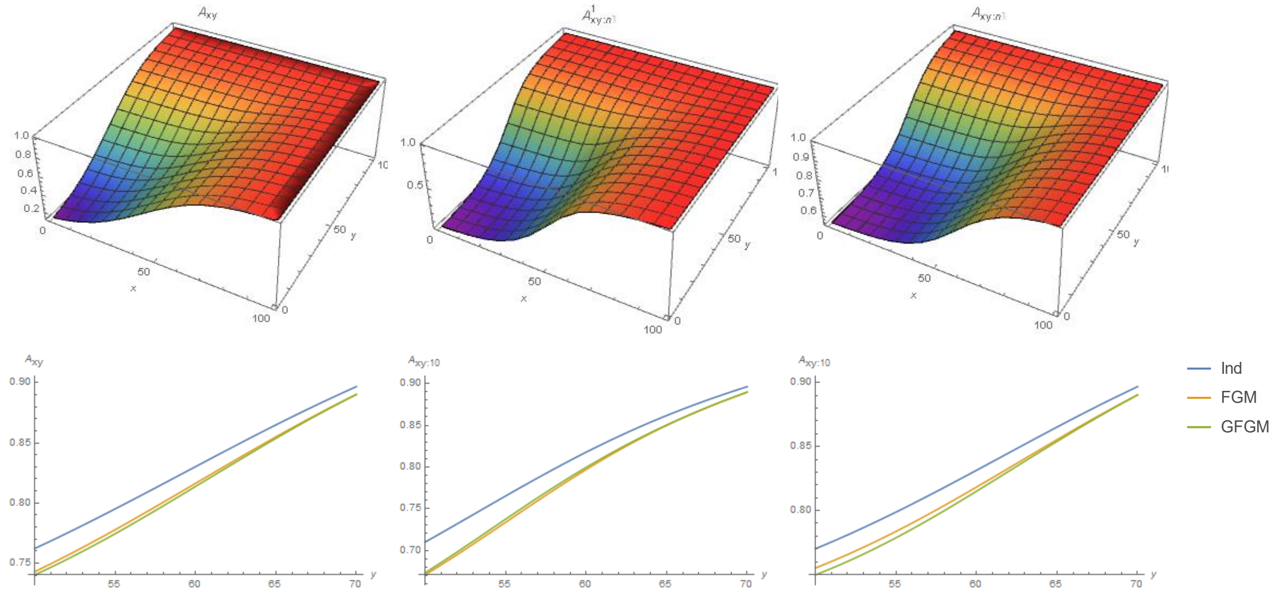

FIGURE 2. (top panel) The 3D plots for the GFGM model and (bottom panel) the plots of $\overline{A}_{xy}$, $\overline{A}^{1}_{xy:n]}$ and $\overline{A}_{xy:n]}$ for the proposed models (ρ = 0.3 and n = 10, x = 50 < y)

THE ILLUSTRATIVE EXAMPLES

This section shows that the premiums have considerable differences due to the asymmetry, even if the insurance policies are issued for ages with the same average. Also, premiums are sensitive to the correlation size. To explain these differences, the net single premiums are computed for various values of ρ and average age 55. Table 3 gives the numerical results based on Gompertz's laws.

TABLE 3. The APVs of the joint-life insurance policies

Spearman's correlations	Age	(x=50,y=60)			(x=55,y=55)			(x=60,y=50)		
	Premiums	Ind	FGM	GFGM	Ind	FGM	GFGM	Ind	FGM	GFGM
ρ = 0.1	$\overline{A}_{xy}$	0.83026	0.82563	0.82477	0.81961	0.81398	0.81345	0.83026	0.82563	0.82640
	$\overline{A}^{1}_{xy:n]}$	0.81772	0.81080	0.81164	0.80265	0.79368	0.79572	0.81772	0.81080	0.81365
	$\overline{A}_{xy:n]}$	0.83148	0.82710	0.82602	0.82140	0.81616	0.81528	0.83148	0.82710	0.82763
ρ = 0.2	$\overline{A}_{xy}$	0.83026	0.82100	0.81927	0.81961	0.80834	0.80728	0.83026	0.82100	0.82253
	$\overline{A}^{1}_{xy:n]}$	0.81772	0.80388	0.80555	0.80265	0.78471	0.78879	0.81772	0.80388	0.80957
	$\overline{A}_{xy:n]}$	0.83148	0.82273	0.82056	0.82140	0.81093	0.80915	0.83148	0.82273	0.82377
ρ = 0.3	$\overline{A}_{xy}$	0.83026	0.81637	0.81378	0.81961	0.80270	0.80111	0.83026	0.81637	0.81867
	$\overline{A}^{1}_{xy:n]}$	0.81772	0.79695	0.79947	0.80265	0.77574	0.78186	0.81772	0.79695	0.80550
	$\overline{A}_{xy:n]}$	0.83148	0.81836	0.81510	0.82140	0.80570	0.80303	0.83148	0.81836	0.81992

To illustrate the effect of asymmetry on premiums according to the various policy types, the results for life insurances in joint life status are exemplified below by assuming ρ = 0.3. Here, the death benefits are considered to be 100.000 units. The net single premiums are denoted by (P^{Ind}, P^{FGM}, P^{GFGM}) for independence, FGM and GFGM models, respectively.

Example 1. For a whole life insurance policy, issued for two individuals ages 50 and 60, the premiums reduce 1389 (= P^{Ind} - P^{FGM} = 83026 - 81637) units with the effect of symmetry and reduce more 259 (= P^{FGM} - P^{GFGM} = 81637 - 81378) units with the effect of asymmetry. Similarly, due to the asymmetric effect, there is a more 252 and 326 unit decrease in premiums according to symmetric impact, respectively, for ten year term and endowment insurance policies.

Example 2. The premiums vary due to the policy types and age averages by assuming {50,60} and ρ = 0.3.

Case ($x < y$): P^{GFGM} is the lowest for whole life (83026, 81637, 81378) and endowment life (83148, 81836, 81510) insurance policies, while P^{FGM} is the lowest in term life (81772, 79695, 79947) insurance policies.

Case (x=y=55): P^{GFGM} is the lowest for whole life (81961, 80270, 80111) and endowment life insurance policies (82140, 80570, 80303), while P^{FGM} is the lowest in term life insurance policies (80265, 77574, 78186).

Case ($x > y$): P^{FGM} is the lowest for whole life (83026, 81637, 81867), term life (81772, 79695, 80550), and endowment life (83148, 81836, 81992) insurance policies.

The examples show that the premiums are decreasing due to asymmetry. The premium change rates can also explain the effect on premiums of asymmetry. Table 4 presents the results and calculated with $P_{ij} = \frac{(P^i - P^j)}{P^i}$ for (i,j) {independent:1, FGM:2, GFGM:3}. Kara (2021) performs similar evaluations for the last survivor-life insurance policies. This paper, differently, summarizes the results for the first death - life insurance policies as follows:

Corollary 4.1 *The insurance policies which are the most affected by asymmetry are term life (3.64%), followed by whole life (2.59%), and finally endowment life (2.51%) policies for and ages (50,55).*

Corollary 4.2 *P_{13} is higher than P_{12} for whole and endowment life insurance policies, while vice versa for term life insurance policies.*

Corollary 4.3 *The ordering of premiums differs due to the same average ages. In the whole and endowment life insurance policies, the direction of variation is positive ($P_{23} > 0$), while in the term insurance policy, it is negative ($P_{23} < 0$) for almost every $x \leq y$. For nearly every $x > y$, it is negative ($P_{23} < 0$) for all insurance policies.*

Corollary 4.4 *The symmetric and asymmetric effects decrease as the age difference increases in all life insurance policy types.*

TABLE 4. The effect of the symmetry and asymmetry on the premiums due to the age differences

Spearman's correlations	Age	(x=50,y=55)			(x=50,y=60)			(x=50,y=65)		
	Premiums	P_{12}	P_{23}	P_{13}	P_{12}	P_{23}	P_{13}	P_{12}	P_{23}	P_{13}
ρ = 0.1	$\bar{A}_{xy}$	0.0073	0.0014	0.0086	0.0056	0.0010	0.0066	0.0038	0.0005	0.0043
	$\bar{A}^{1}_{xy:n]}$	0.0138	-0.0017	0.0121	0.0085	-0.0010	0.0074	0.0046	-0.0002	0.0044
	$\bar{A}_{xy:n]}$	0.0063	0.0021	0.0084	0.0053	0.0013	0.0066	0.0038	0.0005	0.0043
ρ = 0.2	$\bar{A}_{xy}$	0.0145	0.0028	0.0173	0.0112	0.0021	0.0132	0.0077	0.0010	0.0086
	$\bar{A}^{1}_{xy:n]}$	0.0275	-0.0034	0.0242	0.0169	-0.0021	0.0149	0.0092	-0.0004	0.0088
	$\bar{A}_{xy:n]}$	0.0126	0.0042	0.0167	0.0105	0.0026	0.0131	0.0075	0.0011	0.0086
ρ = 0.3	$\bar{A}_{xy}$	0.0218	0.0043	0.0259	0.0167	0.0032	0.0198	0.0115	0.0015	0.0129
	$\bar{A}^{1}_{xy:n]}$	0.0413	-0.0051	0.0364	0.0254	-0.0032	0.0223	0.0138	-0.0006	0.0132
	$\bar{A}_{xy:n]}$	0.0189	0.0063	0.0251	0.0158	0.0040	0.0197	0.0113	0.0016	0.0129

Conclusions

In this paper, we investigated the problem of pricing for life insurance policies using the symmetric and asymmetric dependence approach under the joint-life status. It is assumed that symmetric and asymmetric dependence lifetimes are modeled with FGM and GFGM Type II copulas. The joint survival and mortality functions are derived for the proposed models under Gompertz marginals. Then, net single premiums are compared for various Spearman correlation values and ages. The results demonstrate that premiums are sensitive to dependency and age differences. It is also observed that the proposed asymmetric model generally has the lowest premium ordering. These results will influence other significant actuarial decisions such as determining investment and reserve amounts.

The fact that premiums are low in dependent and asymmetric models can be interpreted as follows for the insured. The surviving spouses' lifetime, who may experience broken heart attack syndrome due to a reason such as the death of one of the spouses from any disease, may change. The spouses' lifetime might be asymmetrically dependent since the spouses are not affected by these events at the same rate. In this study, in such cases, purchasing low-premium insurance products with the assumed asymmetric model may be an incentive for such cases. For example, given the future lifetime of individuals aged 60 and over who are most affected by a current pandemic, insured persons may be advised to purchase lower-priced dependent insurance products for their surviving spouses.

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APPENDIX

Here, the proofs of lemmas 5.1 and 5.2 are given using the survival and mortality functions for the independence model by ${}_tp_{xy} = {}_tp_{x\,t}p_y$ and $\mu_{x+t:y+t} = \mu_{x+t} + \mu_{y+t}$. The functions are ${}_tp_{xy} = e^{-\frac{B(-1+c^t)(c^x+c^y)}{\mathrm{Log}[c]}}$ and ${}_t\mu_{xy} = Bc^t\,(c^x+c^y)$ for Gompertz marginals: $= e^{-\left(\frac{Bc^x(-1+c^t)}{Log[c]}\right)}$ and ${}_t\mu_x = Bc^{x+t}$.

Proof of Lemma 4.1 The survival and mortality functions are derived for the FGM copula.

The joint survival function for FGM has the explicit Mathematica solution by Equation (7) as follow:

$$ {}_tp_{xy} = Ke^{-\frac{2B(-1+c^t)(c^x+c^y)}{\mathrm{Log}[c]}} \tag{A1} $$

where $K = \theta - e^{\frac{Bc^x(-1+c^t)}{Log[c]}}\theta - e^{\frac{Bc^y(-1+c^t)}{Log[c]}}\theta + e^{\frac{B(-1+c^t)(c^x+c^y)}{Log[c]}}(1+\theta)$ or it can be re-written in terms of ${}_tp_x$ and ${}_tp_y$

$$ K = \theta - \frac{1}{{}_tp_x}\theta - \frac{1}{{}_tp_y}\theta + \frac{1}{{}_tp_{x\,t}p_y}(1+\theta) $$

$$ = \frac{1}{{}_tp_{x\,t}p_y}\left(1+\theta\left(1 - {}_tp_x - {}_tp_y + {}_tp_{x\,t}p_y\right)\right) $$

by substituting the K and ${}_tp_x$ in Equation (A1)

$$ {}_tp_{xy} = K\,{}_tp_{x\,t}p_y $$

$$ = \frac{1}{{}_tp_{x\,t}p_y}\left(1+\theta\left(1 - {}_tp_x - {}_tp_y + {}_tp_{x\,t}p_y\right)\right)\ {}_tp_{x\,t}p_y $$

$$ = \left(1+\theta\left(1 - {}_tp_x - {}_tp_y + {}_tp_{x\,t}p_y\right)\right) $$

The joint mortality function is derived by Mathematica solution using Equation (6) as follows;

$$ {}_t\mu_{xy} = \frac{1}{K}\left(Bc^t\left(c^x\left(\theta - e^{\frac{Bc^y(-1+c^t)}{\mathrm{Log}[c]}}\theta + K\right) + c^y\left(\theta - e^{\frac{Bc^x(-1+c^t)}{\mathrm{Log}[c]}}\theta + K\right)\right)\right) $$

$$ = \frac{1}{K}\left(Bc^tc^x\left(\theta - e^{\frac{Bc^y(-1+c^t)}{\mathrm{Log}[c]}}\theta + K\right) + Bc^tc^y\left(\theta - e^{\frac{Bc^x(-1+c^t)}{\mathrm{Log}[c]}}\theta + K\right)\right) $$

$$ = \frac{1}{K}\left({}_t\mu_x\left(\theta - \left(\frac{1}{{}_tp_y}\right)\theta + K\right) + {}_t\mu_y\left(\theta - \left(\frac{1}{{}_tp_x}\right)\theta + K\right)\right) $$

$$ = \frac{1}{K}\left({}_t\mu_x\left(K - \frac{{}_tq_y}{{}_tp_y}\theta\right) + {}_t\mu_y\left(K - \frac{{}_tq_x}{{}_tp_x}\theta\right)\right). $$

Proof of Lemma 4.2 The survival and mortality functions are derived for the GFGM copula.

The joint survival function for GFGM has the explicit Mathematica solution by Equation (8) as follow:

$$ {}_tp_{xy} = e^{-\frac{B(-1+c^t)(c^x+c^y)}{\mathrm{Log}[c]}} + G \tag{A2} $$

where $G = \left(e^{-\frac{Bc^x(-1+c^t)}{\mathrm{Log}[c]}}\right)^\alpha \left(e^{-\frac{Bc^y(-1+c^t)}{\mathrm{Log}[c]}}\right)^\beta \left(1 - e^{-\frac{Bc^x(-1+c^t)}{\mathrm{Log}[c]}}\right)^b \left(1 - e^{-\frac{Bc^y(-1+c^t)}{\mathrm{Log}[c]}}\right)^b \theta$ or it is formed as $G = \theta({}_tp_x)^\alpha({}_tp_y)^\beta(1 - {}_tp_x)^b(1 - {}_tp_y)^b$. On the other hand, by substituting the G and ${}_tp_x$ in Equation (A2)

$$ {}_tp_{xy} = e^{-\frac{B(-1+c^t)(c^x+c^y)}{\mathrm{Log}[c]}} + G $$

$$ = {}_tp_{x\,t}p_y + ({}_tp_x)^\alpha({}_tp_y)^\beta(1 - {}_tp_x)^b(1 - {}_tp_y)^b\theta $$

$$ = {}_tp_{x\,t}p_y + \theta({}_tp_x)^\alpha({}_tp_y)^\beta({}_tq_{x\,t}q_y)^b $$

The joint mortality function is derived by Mathematica solution using Equation (6) as follows;

$$\mu_{xy} = -\left(\frac{1}{e^{-\frac{B(-1+c^t)(c^x+c^y)}{\mathrm{Log}[c]}} + G}\right)(Bc^t(c^x(-e^{-\frac{B(-1+c^t)(c^x+c^y)}{\mathrm{Log}[c]}}$$

$$+\frac{bG}{-1+e^{\frac{Bc^x(-1+c^t)}{\mathrm{Log}[c]}}} - \alpha G)$$

$$+c^y(-e^{-\frac{B(-1+c^t)(c^x+c^y)}{\mathrm{Log}[c]}} + \frac{bG}{-1+e^{\frac{Bc^y(-1+c^t)}{\mathrm{Log}[c]}}} - \beta G))$$

$$= -\left(\frac{1}{{}_tp_{x\,t}p_y + G}\right)(Bc^{x+t}(-{}_tp_{x\,t}p_y + \frac{bG}{-1+1/{}_tp_x} - \alpha G)$$

$$+ Bc^{y+t}(-{}_tp_{x\,t}p_y + \frac{bG}{-1+1/{}_tp_y} - \beta G))$$

$$= \left(\frac{1}{{}_tp_{x\,t}p_y + G}\right)({}_t\mu_x({}_tp_{x\,t}p_y - G\left(\frac{b_t p_x}{{}_tq_x} - \alpha\right)) - {}_t\mu_y({}_tp_{x\,t}p_y$$

$$- G\left(\frac{b_t p_y}{{}_tq_y} - \beta\right)))$$

$$= \left(\frac{1}{{}_tp_{x\,t}p_y + G}\right)({}_tp_{x\,t}p_y({}_t\mu_x - {}_t\mu_y) - bG(\left(\frac{{}_tp_x}{{}_tq_x}\right)_t \mu_x$$

$$- bG(\left(\frac{{}_tp_x}{{}_tq_x}\right)_t \mu_x - \left(\frac{{}_tp_y}{{}_tq_y}\right)_t \mu_y) - G(\alpha_t\mu_x + \beta_t\mu_y)).$$