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Intuitionistic Anti Fuzzy Normal Subrings over Normed Rings

(Subgelang Normal Kabur Anti Berintuisi terhadap Gelang Norma)

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ABSTRACT

In this paper, we initiate the notion of intuitionistic anti fuzzy normed normal subrings and generalize various related properties. We extend the notion of intuitionistic fuzzy normed subrings to intuitionistic anti fuzzy normed normal subrings. Further, we study the algebraic nature of direct product of intuitionistic anti fuzzy normed normal subrings and establish and examine some imperative properties of such products. We also provide some essential operations specially subset, complement and intersection relating to direct product of intuitionistic anti fuzzy normed normal subrings. Then, we generalize the relation between the intuitionistic characteristic function and direct product of intuitionistic anti fuzzy normed normal subrings.

Keywords: Direct product of intuitionistic anti fuzzy normed normal subrings; intuitionistic anti fuzzy normed normal subring; intuitionistic fuzzy normed normal subring

ABSTRAK

Dalam makalah ini, kami mempelopori tanggapan subgelang normal norma anti kabur berintuisi dan menyamaratakan pelbagai sifat berkaitan. Kami kembangkan tanggapan subgelang normal norma kabur berintuisi kepada subgelang normal norma anti kabur berintuisi. Seterusnya, kami mengkaji sifat aljabar bagi hasil darab langsung subgelang normal norma anti kabur berintuisi dan memantapkan serta mengkaji beberapa sifat penting hasil darab tersebut. Kami juga memberikan beberapa operasi asas terutamanya subset, pelengkap dan persilangan yang berkaitan dengan hasil darab langsung subgelang normal norma anti kabur berintuisi. Kemudian kami permudahkan hubungan antara fungsi cirian berintuisi dengan hasil darab langsung subgelang normal norma anti kabur berintuisi.

Kata kunci: Hasil darab langsung subgelang normal bernorma anti kabur berintuisi; subgelang normal bernorma anti kabur berintuisi; subgelang normal bernorma kabur berintuisi

INTRODUCTION

The fundamental concept of fuzzy sets was initiated by Zadeh (1965). After that, many researchers applied this concept in many other branches. Rosenfeld (1971) studied fuzzy group theory by introducing the concepts of fuzzy subgroupoid and fuzzy subgroup. Later, Liu (1982) introduced the notion of fuzzy ring and discussed fuzzy subrings and fuzzy ideals and presented some basic concepts of fuzzy algebra, as fuzzy invariant subgroups, fuzzy ideals and proved some related properties. The studies of anti fuzzy subgroups of groups was introduced by Biswas (1990). He showed that a fuzzy subset of a group is a fuzzy subgroup if and only if the complement of the fuzzy subset is anti fuzzy subgroup. Azam et al. (2013) studied anti fuzzy rings and presented the notion of anti fuzzy ideals of a ring and discussed some of its properties. Rasuli (2019) defined anti fuzzy subrings by using t-conorm and considered the properties of intersection and direct product and homomorphisms for anti fuzzy subrings with respect to t-conorm.

The notion of intuitionistic fuzzy set was introduced by Atanassov (1986), as a generalization of fuzzy sets. Li et al. (2009), defined for the first time the intuitinoistic anti fuzzy subgroup and intuitionistic anti fuzzy normal subgroup and some important conclusions were presented. In 2012, Sharma and Bansal introduced the notion of intuitionistic anti fuzzy subrings and ideals in a ring and studied their properties. Later, Anitha (2019) introduced some properties of intuitionistic anti fuzzy normal subrings and discussed direct product of intuitionistic anti fuzzy normal subrings. Kausar (2019) investigated the concept of intuitionistic anti fuzzy normal subrings over non-associative rings and gave some properties of such subrings and defined the direct product of finite intuitionistic anti fuzzy subrings.

The aim of this paper is to introduce the notion of intuitionistic anti fuzzy normed normal subrings. Also, to prove that the intersection and direct product of two intuitionistic anti fuzzy normed normal subring is an intuitionistic anti fuzzy normed normal subring and to define the relationship between intuitionistic anti characteristic function and intuitionistic anti fuzzy normed normal subring. Finally, we obtain some results for direct product of intuitionistic anti fuzzy normed normal subrings and various fundamental properties will be examined.

PRELIMINARIES

In this section, we recall some of the fundamental and significant definitions and results required for the following sections.

Definition 1 (Naimark 1964) A ring R is said to be a normed ring *(NR)* if R possesses a norm $\| \ \|$, that is, a non-negative real-valued function $\| \ \| : R \to R$ such that for any $v, r \in R$,

1. $\| v \| = 0 \Leftrightarrow v = 0$,
2. $\| v + r \| \leq \| v \| + \| r \|$,
3. $\| v \| = \| -v \|$, and
4. $\| vr \| \leq \| v \| \| r \|$.

Definition 2 (Al-Masarwah & Ahmad 2020) Let $*:[0,1]\times[0,1]\to[0,1]$ be a binary operation. Then * is a *t*-norm if * satisfies the conditions of commutativity, associativity, monotonicity and neutral element *1.* We shortly use *t*-norm and write $v*r$ instead of $*(v,r)$.

Definition 3 (Gupta & Qi 1991) Let $\diamond:[0,1]\times[0,1]\to[0,1]$ be a binary operation. Then $\diamond$ is a *s*-norm if $\diamond$ satisfies the conditions of commutativity, associativity, monotonicity and neutral element *0.* We shortly use *s*-norm and write $v \diamond r$ instead of $\diamond(v,r)$.

Definition 4 (Ahmad & Hasan 2011a) The fuzzy set *(FS)* A on a universal X is a set of ordered pairs:

$$A = \{(v, \mu_A(v)) : v \in X\},$$

where, $\mu_A(v)$ is the membership function of v in A. For all $v \in X$, we have $0 \leq \mu_A(v) \leq 1$.

Let A be a fuzzy set defined in X. The support of *FS A* is the crisp set of all elements in X such that the membership function of A is non-zero, that is, *supp* $(A) = \{v \in X \mid \mu_A(v) > 0\}$ (Ahmad & Hasan 2011b).

Definition 5 (Alsarahead & Ahmad 2018) An intuitionistic fuzzy set (*IFS*) A in a nonempty set X is an object having the form *IFS* $A=\{(v, \mu_A(v), \gamma_A(v)) : v \in X\}$, where the functions $\mu_A(v): X \to [0,1]$ and $\gamma_A(v): X \to [0,1]$ denote the degree of membership and the degree of nonmembership, respectively, where $0 \leq \mu_A(v) + \gamma_A(v) \leq 1$ for all $v \in X$. An intuitionistic fuzzy set A is written symbolically in the form $A = (\mu_A, \gamma_A)$.

The support of an *IFS A* in a universe X is a crisp set that contains all the elements of X that have greater than zero membership values in A and less than one non-membership values in A, that is, *supp* $(A) = \{v \in X \mid \mu_A(v) > 0$ and $\gamma_A(v) < 1\}$ (Marashdeh & Salleh 2011)

Definition 6 (Abed Alhaleem & Ahmad 2020) Let * be a continuous *t*-norm and $\diamond$ be a continuous *s*-norm. An intuitionistic fuzzy set $A = \{(v, \mu_A(v), \gamma_A(v)) : v \in NR\}$ is called an intuitionistic fuzzy normed subring *(IFNSR)* of the normed ring $(NR,+,.)$ if it satisfies the following conditions for all $v, r \in NR$:

i. $\mu_A(v - r) \geq \mu_A(v) * \mu_A(r)$,

ii. $\mu_A(vr) \geq \mu_A(v) * \mu_A(r)$,

iii. $\gamma_A(v - r) \leq \gamma_A(v) \diamond \gamma_A(r)$,

iv. $\gamma_A(vr) \leq \gamma_A(v) \diamond \gamma_A(r)$.

Definition 7 (Abed Alhaleem & Ahmad 2021) Let *NR* be a normed ring. An intuitionistic fuzzy subring A of *NR* is said to be an intuitionistic fuzzy normed normal subring *(IFNNSR)* of *NR* if it satisfies the following for all $v, r \in NR$:

i. $\mu_A(vr) = \mu_A(rv)$,

ii. $\gamma_A(vr) = \gamma_A(rv)$.

SOME PROPERTIES OF INTUITIONISTIC ANTI FUZZY NORMED NORMAL SUBRINGS

In this section, we introduce the notion of intuitionistic anti fuzzy normed normal subrings and present relevant related properties.

Definition 8 Let $*$ be a continuous *t*-norm and $\diamond$ be a continuous *s*-norm. An intuitionistic fuzzy set $A=\{(v, \mu_A(v), \gamma_A(v)): v \in NR\}$ is said to be an intuitionistic anti fuzzy normed subring (*IAFNNSR*) of the normed ring $(NR,+,.)$ if it satisfies the following for all $v, r \in NR$:

i. $\mu_A(v-r) \leq \mu_A(v) \diamond \mu_A(r)$,

ii. $\mu_A(vr) \leq \mu_A(v) \diamond \mu_A(r)$,

iii. $\gamma_A(v-r) \geq \gamma_A(v) * \gamma_A(r)$,

iv. $\gamma_A(vr) \geq \gamma_A(v) * \gamma_A(r)$.

Definition 9 Let *NR* be a normed ring. An intuitionistic anti fuzzy subring *A* of *NR* is said to be an intuitionistic anti fuzzy normed normal subring *(IAFNNSR)* of *NR* if it satisfies the following for all $v, r \in NR$:

i. $\mu_A(vr) = \mu_A(rv)$,

ii. $\gamma_A(vr) = \gamma_A(rv)$.

Proposition 1 Let $(NR,+,.)$ be a normed ring. If A and B are two intuitionistic anti fuzzy normed normal subrings of *NR,* then their intersection $(A \cap B)$ is an intuitionistic anti fuzzy normed normal subring of *NR*.

Proof Let $A=\{(v,\mu_A(v),\gamma_A(v)): v \in NR\}$ and $B = \{(v, \mu_B(v), \gamma_B(v)): v \in NR\}$ be intuitionistic anti fuzzy normed normal subrings. Let $C = A \cap B$ such that $C = \{(c,\mu_C(c), \gamma_C(c)): c \in NR\}$ where $\mu_C(c) = max\{\mu_A(c), \mu_B(c)\}$ and $\gamma_C(c) = min\{\gamma_A(c), \gamma_B(c)\}$. Let $v, r \in NR,$ then

$$\begin{aligned} \mu_C(v-r) &= max\{\mu_A(v-r), \mu_B(v-r)\} \\ &= \mu_A(v-r) \diamond \mu_B(v-r) \\ &\leq \{\mu_A(v) \diamond \mu_A(r)\} \diamond \{\mu_B(v) \diamond \mu_B(r)\} \\ &= \mu_A(v) \diamond \{\mu_A(r) \diamond \mu_B(v)\} \diamond \mu_B(r) \\ &= \mu_A(v) \diamond \{\mu_B(v) \diamond \mu_A(r)\} \diamond \mu_B(r) \\ &= \{\mu_A(v) \diamond \mu_B(v)\} \diamond \{\mu_A(r) \diamond \mu_B(r)\} \\ &= \mu_C(v) \diamond \mu_C(r) \end{aligned}$$

and

$$\begin{aligned} \mu_C(vr) &= max\{\mu_A(vr), \mu_B(vr)\} \\ &= \mu_A(vr) \diamond \mu_B(vr) \\ &\leq \{\mu_A(v) \diamond \mu_A(r)\} \diamond \{\mu_B(v) \diamond \mu_B(r)\} \\ &= \mu_A(v) \diamond \{\mu_A(r) \diamond \mu_B(v)\} \diamond \mu_B(r) \\ &= \mu_A(v) \diamond \{\mu_B(v) \diamond \mu_A(r)\} \diamond \mu_B(r) \\ &= \{\mu_A(v) \diamond \mu_B(v)\} \diamond \{\mu_A(r) \diamond \mu_B(r)\} \\ &= \mu_C(v) \diamond \mu_C(r). \end{aligned}$$

Similarly

$\gamma_C(v-r) \geq \gamma_C(v) * \gamma_C(r)$

and

$\gamma_C(vr) \geq \gamma_C(v) * \gamma_C(r)$.

Thus *C* is an intuitionistic anti fuzzy normed subring of *NR*. Now,

$$\begin{aligned} \mu_C(vr) &= \mu_A(vr) \diamond \mu_B(vr) \\ &= \mu_A(rv) \diamond \mu_B(rv) \\ &= \mu_C(rv). \end{aligned}$$

Therefore $\mu_C(vr) = \mu_C(rv)$.

Also,

$$\begin{aligned} \gamma_C(vr) &= \gamma_A(vr) * \gamma_B(vr) \\ &= \gamma_A(rv) * \gamma_B(rv) \\ &= \gamma_C(rv). \end{aligned}$$

Therefore $\gamma_C(vr) = \gamma_C(rv)$.

Then, the intersection of any two intuitionistic anti fuzzy normed normal subrings is an intuitionistic anti fuzzy normed normal subring of *NR*.

Definition 10 Let *A* be a non-empty subset of the normed ring *NR,* the intuitionistic anti characteristic function of *A* is defined as $\lambda_A = (\mu_{\lambda_A}, \gamma_{\lambda_A})$, where

$$\mu_{\lambda_A}(v) = \begin{cases} 0, & if \ v \in A \\ 1, & if \ v \notin A \end{cases} \quad \text{and} \quad \gamma_{\lambda_A}(v) = \begin{cases} 1, & if \ v \in A \\ 0, & if \ v \notin A. \end{cases}$$

Lemma 1 If $A = (\mu_A, \gamma_A)$ is a subring of *NR* then $\lambda_A = (\mu_{\lambda_A}, \gamma_{\lambda_A})$ is an intuitionistic anti fuzzy normed normal subring of *NR*.

Proof Let $v, r \in A$, then by definition of the intuitionistic anti characteristic function $\mu_{\lambda_A}(v) = 0 = \mu_{\lambda_A}(r)$ and $\gamma_{\lambda_A}(v) = 1 = \gamma_{\lambda_A}(r)$. Since $v - r$, vr in *A*, then, $\mu_{\lambda_A}(v - r) = 0 = 0 \diamond 0 = \mu_{\lambda_A}(v) \diamond \mu_{\lambda_A}(r)$ and $\mu_{\lambda_A}(vr) = 0 = 0 \diamond 0 = \mu_{\lambda_A}(v) \diamond \mu_{\lambda_A}(r)$. Thus, $\mu_{\lambda_A}(v - r) \leq \mu_{\lambda_A}(v) \diamond \mu_{\lambda_A}(r)$ and $\mu_{\lambda_A}(vr) \leq \mu_{\lambda_A}(v) \diamond \mu_{\lambda_A}(r)$. Now $\gamma_{\lambda_A}(v - r) = 1 = 1 * 1 = \gamma_{\lambda_A}(v) * \gamma_{\lambda_A}(r)$ and $\gamma_{\lambda_A}(vr) = 1 = 1 * 1 = \gamma_{\lambda_A}(v) * \gamma_{\lambda_A}(r)$. Thus, $\gamma_{\lambda_A}(v - r) \geq \gamma_{\lambda_A}(v) * \gamma_{\lambda_A}(r)$ and $\gamma_{\lambda_A}(vr) \geq \gamma_{\lambda_A}(v) * \gamma_{\lambda_A}(r)$. As vr and $rv \in$A, so $\mu_{\lambda_A}(vr) = 0 = \mu_{\lambda_A}(rv)$ and $\gamma_{\lambda_A}(vr) = 1 = \gamma_{\lambda_A}(rv)$.

Accordingly, $\mu_{\lambda_A}(vr) = \mu_{\lambda_A}(rv)$ and $\gamma_{\lambda_A}(vr) = \gamma_{\lambda_A}(rv)$. Similarly, we have when $v, r \notin A$:

$$\mu_{\lambda_A}(v - r) \leq \mu_{\lambda_A}(v) \diamond \mu_{\lambda_A}(r),$$
$$\mu_{\lambda_A}(vr) \leq \mu_{\lambda_A}(v) \diamond \mu_{\lambda_A}(r),$$
$$\gamma_{\lambda_A}(v - r) \geq \gamma_{\lambda_A}(v) * \gamma_{\lambda_A}(r),$$
$$\gamma_{\lambda_A}(vr) \geq \gamma_{\lambda_A}(v) * \gamma_{\lambda_A}(r).$$

Also, $\mu_{\lambda_A}(vr) = \mu_{\lambda_A}(rv)$ and $\gamma_{\lambda_A}(vr) = \gamma_{\lambda_A}(rv)$.

Hence, the intuitionistic anti characteristic function $\lambda_A = (\mu_{\lambda_A}, \gamma_{\lambda_A})$ is an intuitionistic anti fuzzy normed normal subring of *NR*.

Lemma 2 If A and B are two subrings of the ring *NR*, then their intersection $A \cap B$ is a subring of *NR* if and only if the intuitionistic anti characteristic function $\lambda_C = (\mu_{\lambda_C}, \gamma_{\lambda_C})$ of $C = A \cap B$ is an intuitionistic anti fuzzy normed normal subring of *NR*.

Proof Let $C = A \cap B$ be a subring of *NR* and $v, r \in NR$. If $v, r \in C$, then by definition of the intuitionistic anti characteristic function $\mu_{\lambda_C}(v) = 0 = \mu_{\lambda_C}(r)$ and $\gamma_{\lambda_C}(v) = 1 = \gamma_{\lambda_C}(r)$. Since $v - r$, vr in A and B, it follows that $v - r$, vr are in C. Thus, $\mu_{\lambda_C}(v - r) = 0 = 0 \diamond 0 = \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r)$ and $\mu_{\lambda_C}(vr) = 0 = 0 \diamond 0 = \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r)$. Thus $\mu_{\lambda_C}(v - r) \leq \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r)$ and $\mu_{\lambda_C}(vr) \leq \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r)$. Now, $\gamma_{\lambda_C}(v - r) = 1 = 1 * 1 = \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r)$ and $\gamma_{\lambda_C}(vr) = 1 = 1 * 1 = \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r)$. Thus, $\gamma_{\lambda_C}(v - r) \geq \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r)$ and $\gamma_{\lambda_C}(vr) \geq \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r)$. As vr and $rv \in C$, so $\mu_{\lambda_C}(vr) = 0 = \mu_{\lambda_C}(rv)$ and $\gamma_{\lambda_C}(vr) = 1 = \gamma_{\lambda_C}(rv)$. Accordingly, $\mu_{\lambda_C}(vr) = \mu_{\lambda_C}(rv)$ and $\gamma_{\lambda_C}(vr) = \gamma_{\lambda_C}(rv)$. Similarly, we have when $v, r \notin C$:,

$$\mu_{\lambda_C}(v - r) \leq \mu_{\lambda_C}(r) \diamond \mu_{\lambda_C}(r),$$
$$\mu_{\lambda_C}(vr) \leq \mu_{\lambda_C}(r) \diamond \mu_{\lambda_C}(r),$$
$$\gamma_{\lambda_C}(v - r) \geq \gamma_{\lambda_C}(r) * \gamma_{\lambda_C}(r),$$
$$\gamma_{\lambda_C}(vr) \geq \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r).$$

Also, $\mu_{\lambda_C}(vr) = \mu_{\lambda_C}(rv)$ and $\gamma_{\lambda_C}(vr) = \gamma_{\lambda_C}(rv)$.

Hence, the intuitionistic anti characteristic function $\lambda_C = (\mu_{\lambda_C}, \gamma_{\lambda_C})$ of C is an intuitionistic anti fuzzy normed normal subring of *NR*.

Conversely, assume that the intuitionistic anti characteristic function $\lambda_C = (\mu_{\lambda_C}, \gamma_{\lambda_C})$ of C is an intuitionistic anti fuzzy normal normed subring of *NR*. Let $v, r \in C$, this imply that $\mu_{\lambda_C}(v) = 0 = \mu_{\lambda_C}(r)$ and $\gamma_{\lambda_C}(v) = 1 = \gamma_{\lambda_C}(r)$, then:

$$\mu_{\lambda_C}(v - r) \leq \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r) = 0 \diamond 0 = 0,$$
$$\mu_{\lambda_C}(vr) \leq \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r) = 0 \diamond 0 = 0,$$
$$\gamma_{\lambda_C}(v - r) \geq \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r) = 1 * 1 = 1,$$
$$\gamma_{\lambda_C}(vr) \geq \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r) = 1 * 1 = 1.$$

This implies that $\mu_{\lambda_C}(v - r) = 0$, $\mu_{\lambda_C}(vr) = 0$ and $\gamma_{\lambda_C}(v - r) = 1$, $\gamma_{\lambda_C}(vr) = 1$. Thus, $v - r$ and $vr \in C$. Hence C is a subring of *NR*.

Proposition 2 If A is an intuitionistic anti fuzzy normed normal subring of a ring *NR*. Then $\triangle A = (\mu_A, \mu_A^c)$ is an intuitionistic anti fuzzy normed normal subring of a ring *NR*.

Proof Let $v, r \in NR$

$$\begin{aligned} \mu_A^c(v - r) &= 1 - \mu_A(v - r) \\ &\geq 1 - (\mu_A(v) \diamond \mu_A(r)) \\ &\geq 1 - max\{\mu_A(v), \mu_A(r)\} \\ &= min\{1 - \mu_A(v), 1 - \mu_A(r)\} \\ &= min\{\mu_A^c(v), \mu_A^c(r)\}. \end{aligned}$$

Then, $\mu_A^c(v - r) \geq \mu_A^c(vr) * \mu_A^c(r)$.

$$\begin{aligned} \mu_A^c(vr) &= 1 - \mu_A(vr) \\ &\geq 1 - (\mu_A(v) \diamond \mu_A(r)) \\ &\geq 1 - max\{\mu_A(v), \mu_A(r)\} \\ &= min\{1 - \mu_A(v), 1 - \mu_A(r)\} \\ &= min\{\mu_A^c(v), \mu_A^c(r)\}. \end{aligned}$$

Then, $\mu_A^c(vr) \geq \mu_A^c(v) * \mu_A^c(r)$.

Also, $\mu_A^c(vr) = 1 - \mu_A(vr) = 1 - \mu_A(rv) = \mu_A^c(rv)$, then $\mu_A^c(vr) = \mu_A^c(rv)$.

Therefore, $\triangle A = (\mu_A, \mu_A^c)$ is an intuitionistic anti fuzzy normed normal subring of *NR*.

Proposition 3 If A is an intuitionistic anti fuzzy normed normal subring of a ring *NR*. Then, $\Diamond A = (\gamma_A^c, \gamma_A)$ is an intuitionistic anti fuzzy normed normal subring of a ring *NR*.

Proof Let $v, r \in NR$

$$\begin{aligned} \gamma_A^c(v - r) &= 1 - \gamma_A(v - r) \\ &\leq 1 - (\gamma_A(v) * \gamma_A(r)) \\ &\leq 1 - min\{\gamma_A(v), \gamma_A(r)\} \\ &= max\{1 - \gamma_A(v), 1 - \gamma_A(r)\} \\ &= max\{\gamma_A^c(v), \gamma_A^c(r)\}. \end{aligned}$$

Then, $\gamma_A^c(v - r) \leq \gamma_A^c(v) \diamond \gamma_A^c(r)$.

$$\begin{aligned} \gamma_A^c(vr) &= 1 - \gamma_A(vr) \\ &\leq 1 - (\gamma_A(v) * \gamma_A(r)) \\ &\leq 1 - min\{\gamma_A(v), \gamma_A(r)\} \\ &= max\{1 - \mu_A(v), 1 - \gamma_A(r)\} \\ &= max\{\gamma_A^c(v), \gamma_A^c(r)\}. \end{aligned}$$

Then, $\gamma_A^c(vr) \leq \gamma_A^c(v) \diamond \gamma_A^c(r)$.

Also, γ_A^c (vr) = 1-γ_A (vr) = 1-γ_A (rv) = γ_A^c (rv), then γ_A^c (vr) = γ_A^c (rv).

Therefore, ◊A = (γ_A^c, γ_A) is an intuitionistic anti fuzzy normed normal ideal of *NR*.

Proposition 4 If *A* is an intuitionistic anti fuzzy normed normal subring of a ring *NR*. Then *A* = *(*μ_A, γ_A*)* is an intuitionistic anti fuzzy normed normal subring of *NR* if the anti fuzzy subsets μ_A and γ_A^c are intuitionistic anti fuzzy normed normal subrings of *NR*.

Proof Clearly, μ_A is an intuitionistic anti fuzzy normed normal subring of *NR,* we need to show that γ_A is an intuitionistic anti fuzzy normed normal subring of *NR*.

$$\begin{aligned} 1-\gamma_A(v-r) &= \gamma_A^c(v-r) \\ &\le \gamma_A^c(v) \diamond \gamma_A^c(r) \\ &\le max\{\gamma_A^c(v), \gamma_A^c(r)\} \\ &= max\{1-\gamma_A(v), 1-\gamma_A(r)\} \\ &= 1-min\{\gamma_A(v), \gamma_A(r)\}. \end{aligned}$$

Then, γ_A $(v$ - $r) \geq \gamma_A$ (v) * γ_A (r).

$$\begin{aligned} 1-\gamma_A(vr) &= \gamma_A^c(vr) \\ &\le \gamma_A^c(v) \diamond \gamma_A^c(r) \\ &\le max\{\gamma_A^c(v), \gamma_A^c(r)\} \\ &= max\{1-\gamma_A(v), 1-\gamma_A(r)\} \\ &= 1-min\{\gamma_A(v), \gamma_A(r)\}. \end{aligned}$$

Then, $\gamma_A(vr) \geq \gamma_A$ (v) * γ_A (r).

Also, 1-γ_A (vr) = γ_A^c (vr) = γ_A^c (rv) = 1-γ_A (rv). Then, $\gamma_A(vr) = \gamma_A$ (rv).

Hence, A = (μ_A, γ_A) is an intuitionistic anti fuzzy normed normal subring of *NR*.

Proposition 5 If *A* is an intuitionistic anti fuzzy normed normal subring of a ring *NR*. Then *A* = (μ_A, γ_A) is an intuitionistic anti fuzzy normed normal subring of *NR* if the anti fuzzy subsets μ_A^c and γ_A are intuitionistic anti fuzzy normed normal subrings of *NR*.

Proof Clearly, γ_A is an intuitionistic anti fuzzy normed normal subring of *NR*. We need to show that μ_A is an intuitionistic anti fuzzy normed normal subring of *NR*.

$$\begin{aligned} 1-\mu_A(v-r) &= \mu_A^c(v-r) \\ &\ge \mu_A^c(v) * \mu_A^c(r) \\ &\ge min\{\mu_A^c(v), \mu_A^c(r)\} \\ &= min\{1-\mu_A(v), 1-\mu_A(r)\} \\ &= 1-max\{\mu_A(v), \gamma_A(r)\}. \end{aligned}$$

Then, μ_A $(v$ - $r) \leq \mu_A$ $(v) \diamond \mu_A$ (r).

$$\begin{aligned} 1-\mu_A(vr) &= \mu_A^c(vr) \\ &\ge \mu_A^c(v) * \mu_A^c(r) \\ &\ge min\{\mu_A^c(v), \mu_A^c(r)\} \\ &= min\{1-\mu_A(v), 1-\mu_A(r)\} \\ &= 1-max\{\mu_A(v), \mu_A(r)\}. \end{aligned}$$

Then, μ_A $(vr) \leq \mu_A$ $(v) \diamond \mu_A$ (r).

Also, 1-μ_A (vr) = μ_A^c (vr) = μ_A^c (rv) = 1-μ_A (rv). Then, μ_A $(vr) = \mu_A$ (rv)

Hence, A = (μ_A, γ_A) is an intuitionistic anti fuzzy normed normal subring of *NR*.

DIRECT PRODUCT OF INTUITIONISTIC ANTI FUZZY NORMED NORMAL SUBRINGS

In this section, we present direct product of intuitionistic anti fuzzy normed normal subrings. If NR_1, NR_2 are rings, then direct product $NR_1 \times NR_2$ of NR_1 and NR_2 is a normed ring with addition + defined as $(v,r) + (z,d) = (v + z,\ r + d)$ and multiplication ∘ defined as $(v,r) \circ (z,d) = (vz,\ rd)$ for every $(v,r),(z,d)$ in $NR_1 \times NR_2$.

Definition 11 An intuitionistic fuzzy set (*IFS*) $A \times B$ = ($\mu_{A\times B}$, $\gamma_{A\times B}$) of $NR_1 \times NR_2$ is an intuitionistic anti fuzzy normed subring (*IAFNSR*) of $NR_1 \times NR_2$ if for all $v = (v_1, v_2)$ and $r = (r_1, r_2)$ in $NR_1 \times NR_2$, satisfies:

i. $\mu_{A\times B}(v-r) \le \mu_{A\times B}(v) \diamond \mu_{A\times B}(r)$,

ii. $\mu_{A\times B}(vr) \le \mu_{A\times B}(v) \diamond \mu_{A\times B}(r)$,

iii. $\gamma_{A\times B}(v-r) \ge \gamma_{A\times B}(v) * \gamma_{A\times B}(r)$,

iv. $\gamma_{A\times B}(vr) \ge \gamma_{A\times B}(v) * \gamma_{A\times B}(r)$.

Definition 12 An intuitionistic anti fuzzy normed subring $A \times B$ = ($\mu_{A\times B}$, $\gamma_{A\times B}$) of ring $NR_1 \times NR_2$ is an intuitionistic anti fuzzy normed normal subring of $NR_1 \times NR_2$ if for all $v = (v_1, v_2)$ and $r = (r_1, r_2)$ in $NR_1 \times NR_2$:

i. $\mu_{A\times B}(vr) = \mu_{A\times B}(rv)$,

ii. $\gamma_{A\times B}(vr) = \gamma_{A\times B}(rv)$.

Lemma 3 If *A* and *B* are subrings of the rings NR_1 and NR_2, respectively, then $A \times B$ is a subring of the ring $NR_1 \times NR_2$ under the same operations defined in $NR_1 \times NR_2$.

Let A and B be two intuitionistic anti fuzzy normed subsets of NR_1 and NR_2, respectively. The direct product of A and B, is denoted by $A\times B$, and defined as:

$A\times B = \{((v,r), \mu_{A\times B}(v,r), \gamma_{A\times B}(v,r))$: for all $v \in NR_1$ and $r \in NR_2\}$

where $\mu_{A\times B}(v,r) = max\{\mu_A(v), \mu_B(r)\}$ and $\gamma_{A\times B}(v,r) = min\{\gamma_A(v), \gamma_B(r)\}$.

Lemma 4 If A and B are intuitionistic anti fuzzy normed normal subrings of rings NR_1 and NR_2, respectively, then $A\times B$ is also an intuitionistic anti fuzzy normed normal subring $NR_1\times NR_2$.

Proof Since the direct product of A and B is denoted by $A\times B = (\mu_{A\times B}, \gamma_{A\times B})$. Let $(v, r),(z,d)$ be in $NR_1\times NR_2$, then:

$$\begin{aligned}\mu_{A\times B}((v,r)-(z,d)) &= \mu_{A\times B}(v-z, r-d)\\ &= max\{\mu_A(v-z), \mu_B(r-d)\}\\ &= \mu_A(v-z) \diamond \mu_B(r-d)\\ &\leq \{\mu_A(v)\diamond\mu_A(z)\}\diamond\{\mu_B(r)\diamond\mu_B(d)\}\\ &= \mu_A(v)\diamond\{\mu_A(z)\diamond\mu_B(r)\}\diamond\mu_B(d)\\ &= \mu_A(v)\diamond\{\mu_B(r)\diamond\mu_A(z)\}\diamond\mu_B(d)\\ &= \{\mu_A(v)\diamond\mu_B(r)\}\diamond\{\mu_A(z)\diamond\mu_B(d)\}\\ &= \mu_{A\times B}(v,r)\diamond\mu_{A\times B}(z,d)\end{aligned}$$

and

$$\begin{aligned}\mu_{A\times B}((v,r)\circ(z,d)) &= \mu_{A\times B}(vz, rd)\\ &= max\{\mu_A(vz), \mu_B(rd)\}\\ &= \mu_A(vz)\diamond\mu_B(rd)\\ &\leq \{\mu_A(v)\diamond\mu_A(z)\}\diamond\{\mu_B(r)\diamond\mu_B(d)\}\\ &= \mu_A(v)\diamond\{\mu_A(z)\diamond\mu_B(r)\}\diamond\mu_B(d)\\ &= \mu_A(v)\diamond\{\mu_B(r)\diamond\mu_A(z)\}\diamond\mu_B(d)\\ &= \{\mu_A(v)\diamond\mu_B(r)\}\diamond\{\mu_A(z)\diamond\mu_B(d)\}\\ &= \mu_{A\times B}(v,r)\diamond\mu_{A\times B}(z,d).\end{aligned}$$

Therefore, $A\times B$ is an intuitionistic anti fuzzy normed subring of $NR_1\times NR_2$.

Now,

$$\begin{aligned}\mu_{A\times B}((v,r)\circ(z,d)) &= \mu_{A\times B}(vz, rd)\\ &= max\{\mu_A(vz), \mu_B(rd)\}\\ &= max\{\mu_A(zv), \mu_B(dr)\}\\ &= \mu_{A\times B}(zv, dr)\\ &= \mu_{A\times B}((z,d)\circ(v,r)).\end{aligned}$$

Similarly,

$$\gamma_{A\times B}((v,r)-(z,d)) \geq \gamma_{A\times B}(v,r) * \gamma_{A\times B}(z,d),$$
$$\gamma_{A\times B}((v,r)\circ(z,d)) \geq \gamma_{A\times B}(v,r) * \gamma_{A\times B}(z,d).$$

and

$\gamma_{A\times B}((v,r)\circ(z,d)) = \gamma_{A\times B}((z,d)\circ(v,r))$.

Hence, $A\times B$ is an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2$.

Proposition 6 Let A and B be an intuitionistic fuzzy sets of the rings NR_1 and NR_2 with identities 1_{NR_1} and 1_{NR_2}, respectively. If $A\times B$ is an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2$, then at least one of the following must holds:

i. $\mu_A(v) \geq \mu_B(1_{NR_2})$ and $\gamma_A(v) \leq \gamma_B(1_{NR_2})$, for all $v \in NR_1$,

ii. $\mu_B(r) \geq \mu_A(1_{NR_1})$ and $\gamma_B(r) \leq \gamma_A(1_{NR_1})$, for all $r \in NR_2$.

Proof Let $A\times B$ be an intuitionistic anti fuzzy normed subring of $NR_1\times NR_2$, and let the statements (i) and (ii) does not holds, we can find $v \in NR_1$ and $r \in NR_2$ such that:

$\mu_A(v) < \mu_B(1_{NR_2})$, $\gamma_A(v) > \gamma_B(1_{NR_2})$ and $\mu_B(r) < \mu_A(1_{NR_1})$,

$\gamma_B(r) > \gamma_A(1_{NR_1})$.

Thus,

$$\begin{aligned}\mu_{A\times B}(vr) &= max\{\mu_A(v), \mu_B(r)\}\\ &< max\{\mu_A(1_{NR_1}), \mu_B(1_{NR_2})\}\\ &= \mu_{A\times B}(1_{NR_1}, 1_{NR_2})\end{aligned}$$

and

$$\begin{aligned}\gamma_{A\times B}(vr) &= min\{\gamma_A(v), \gamma_B(r)\}\\ &> min\{\gamma_A(1_{NR_1}), \gamma_B(1_{NR_2})\}\\ &= \gamma_{A\times B}(1_{NR_1}, 1_{NR_2})\end{aligned}$$

which implies that $A\times B$ is not an intuitionistic anti fuzzy normed subring of $NR_1\times NR_2$ which a contradiction. Therefore, at least one of the statements must hold.

Lemma 5 Let A and B be an intuitionistic anti fuzzy subsets of the rings NR_1 and NR_2 with identities 1_{NR_1} and 1_{NR_2}, respectively. If $A\times B$ is an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2$, then the following are true:

i. if $\mu_A(v) \geq \mu_B(1_{NR_2})$ and $\gamma_A(v) \leq \gamma_B(1_{NR_2})$, then A is an intuitionistic anti fuzzy normed normal subring of NR_1.

ii. if $\mu_B(r) \geq \mu_A(1_{NR_1})$ and $\gamma_B(r) \leq \gamma_A(1_{NR_1})$, then B is an intuitionistic anti fuzzy normed normal subring of NR_2.

Proof Let $A\times B$ be an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2$ with $v, r \in NR_1$ and $1_{NR2} \in NR_2$. Then $(v, 1_{NR_2})$ and $(r, 1_{NR_2})$ are in $NR_1\times NR_2$. Obviously, A is an intuitionistic anti fuzzy normed subring of NR_1, then

i.

$$\begin{aligned}\mu_A(v-r) &= \mu_A(v+(-r))\\ &= max\{\mu_A(v+(-r)), \mu_B(1_{NR_2}+(-1_{NR_2}))\}\\ &= \mu_{A\times B}((v+(-r)),(1_{NR_2}+(-1_{NR_2}))\\ &= \mu_{A\times B}((v,1_{NR_2})+(-r,-1_{NR_2}))\\ &= \mu_{A\times B}((v,1_{NR_2})-(r,1_{NR_2}))\\ &\leq \mu_{A\times B}(v,1_{NR_2})\diamond\mu_{A\times B}(r,1_{NR_2})\\ &= max\{\mu_A(v),\mu_B(1_{NR_2})\}\diamond max\{\mu_A(r),\mu_B(1_{NR_2})\}\\ &= \mu_A(v)\diamond\mu_A(r).\end{aligned}$$

Also,

$$\begin{aligned}\mu_A(vr) &= max\{\mu_A(vr),\mu_B(1_{NR_2}1_{NR_2})\}\\ &= \mu_{A\times B}(vr,1_{NR_2}1_{NR_2})\\ &= \mu_{A\times B}((v,1_{NR_2})\circ(r,1_{NR_2}))\\ &\leq \mu_{A\times B}(v,1_{NR_2})\diamond\mu_{A\times B}(r,1_{NR_2})\\ &= max\{\mu_A(v),\mu_B(1_{NR_2})\}\diamond max\{\mu_A(r),\mu_B(1_{NR_2})\}\\ &= \mu_A(v)\diamond\mu_A(r).\end{aligned}$$

And with,

$$\begin{aligned}\mu_A(vr) &= max\{\mu_A(vr),\mu_B(1_{NR_2}1_{NR_2})\}\\ &= \mu_{A\times B}((vr),(1_{NR_2}1_{NR_2}))\\ &= \mu_{A\times B}((v,1_{NR_2})\circ(r,1_{NR_2}))\\ &= \mu_{A\times B}((r,1_{NR_2})\circ(v,1_{NR_2}))\\ &= \mu_{A\times B}((rv),(1_{NR_2}1_{NR_2}))\\ &= max\{\mu_A(rv),\mu_B(1_{NR_2}1_{NR_2})\}\\ &= \mu_A(rv).\end{aligned}$$

Similarly, we can prove that $\gamma_A(v - r) \geq \gamma_A(v) * \gamma_A(r)$, $\gamma_A(vr) \geq \gamma_A(v) * \gamma_A(r)$ and $\gamma_A(vr) = \gamma_A(rv)$ for all v, $r \in NR_1$. Hence, A is an intuitionistic anti fuzzy normed normal subring of NR_1.

ii. The proof is similar to the above.

Definition 13 Let $A \times B$ be a non-empty subset of the ring $NR_1 \times NR_2$. The intuitionistic anti characteristic function of $A \times B$ is denoted by $\lambda_{A\times B} = (\mu_{\lambda_{A\times B}}, \gamma_{\lambda_{A\times B}})$ and defined as:

$$\lambda_{A\times B}(v) = \begin{cases} 0, & if\ v \in A\times B\\ 1, & if\ v\notin A\times B\end{cases} \text{ and } \gamma_{A\times B}(v) = \begin{cases} 0, & if\ v \in A\times B\\ 1, & if\ v\notin A\times B\end{cases}$$

Theorem 1 Let A and B be two subrings of the rings NR_1 and NR_2, respectively. Then $A\times B$ is a subring of $NR_1\times NR_2$ if and only if the intuitionistic anti characteristic function $\lambda_C = (\mu_{\lambda_C}, \gamma_{\lambda_C})$ of $C = A\times B$ is an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2$.

Proof Let C = $A\times B$ be a subring of $NR_1\times NR_2$ and $v, r \in NR_1\times NR_2$. If $v, r \in C = A\times B$, then by definition of intuitionistic anti characteristic function $\mu_{\lambda_C}(v) = 0 = \mu_{\lambda_C}(r)$ and $\gamma_{\lambda_C}(v) = 1 = \gamma_{\lambda_C}(r)$. Since $v - r$ and $vr \in C$ and C is a subring. It follows that $\mu_{\lambda_C}(v - r) = 0 = 0 \diamond 0 = \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r)$ and $\mu_{\lambda_C}(vr) = 0 = 0 \diamond 0 = \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r)$.

Thus $\mu_{\lambda_C}(v - r) \leq \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r)$ and $\mu_{\lambda_C}(vr) \leq \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r)$. Now $\gamma_{\lambda_C}(v - r) = 1 = 1 * 1 = \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r)$ and $\gamma_{\lambda_C}(v - r) = 1 = 1 * 1 = \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r)$. Thus, $\gamma_{\lambda_C}(v - r) \geq \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r)$ and $\gamma_{\lambda_C}(vr) \geq \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r)$. As vr and $rv \in C$, so $\mu_{\lambda_C}(vr) = 0 = \mu_{\lambda_C}(rv)$ and $\gamma_{\lambda_C}(vr) = 1 = \gamma_{\lambda_C}(rv)$. This implies that $\mu_{\lambda_C}(vr) = \mu_{\lambda_C}(rv)$ and $\gamma_{\lambda_C}(vr) = \gamma_{\lambda_C}(rv)$. Similarly, when $v, r \notin C$, we have:

$$\mu_{\lambda_C}(v - r) \leq \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r),$$
$$\mu_{\lambda_C}(vr) \leq \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r),$$
$$\gamma_{\lambda_C}(v - r) \geq \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r),$$
$$\gamma_{\lambda_C}(vr) \geq \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r).$$

Also, $\mu_{\lambda_C}(vr) = \mu_{\lambda_C}(rv)$ and $\gamma_{\lambda_C}(vr) = \gamma_{\lambda_C}(rv)$.

Hence, the intuitionistic anti characteristic function $\lambda_C = (\mu_{\lambda_C}, \gamma_{\lambda_C})$ of $C = A\times B$ is an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2$.

On the other hand, assume that the intuitionistic anti characteristic function $\lambda_C = (\mu_{\lambda_C}, \gamma_{\lambda_C})$ of $C = A\times B$ is an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2$. Now we need to show that $C = A\times B$ is a subring of NR. Let $v, r \in C$, where $v = (v_1, v_2)$ and $r = (r_1, r_2)$, where $v_1, r_1 \in A$ and $v_2, r_2 \in B$. By definition the intuitionistic anti characteristic function $\mu_{\lambda_C}(v) = 0 = \mu_{\lambda_C}(r)$ and $\gamma_{\lambda_C}(v) = 1 = \gamma_{\lambda_C}(r)$ then,

$$\mu_{\lambda_C}(v - r) \leq \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r) = 0 \diamond 0 = 0,$$
$$\mu_{\lambda_C}(vr) \leq \mu_{\lambda_C}(v) \diamond \mu_{\lambda_C}(r) = 0 \diamond 0 = 0,$$
$$\gamma_{\lambda_C}(v - r) \geq \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r) = 1 * 1 = 1,$$
$$\gamma_{\lambda_C}(vr) \geq \gamma_{\lambda_C}(v) * \gamma_{\lambda_C}(r) = 1 * 1 = 1.$$

This implies that $\mu_{\lambda_C}(v - r) = 0$, $\mu_{\lambda_C}(vr) = 0$ and $\gamma_{\lambda_C}(v - r) = 1$, $\gamma_{\lambda_C}(vr) = 1$. Thus $v - r$ and $vr \in C$.

Hence $C = A\times B$ is a subring of $NR_1\times NR_2$.

Lemma 6 If $V = A\times B$ and $Q = C\times D$ are two subrings of $NR_1\times NR_2$, then their intersection $V \cap Q$ is also a subring of $NR_1\times NR_2$.

Theorem 2 Let $V = A\times B$ and $Q = C\times D$ be two intuitionistic anti fuzzy normed subrings of $NR_1 \times NR_2$. Then $V \cap Q$ is subring of $NR_1 \times NR_2$ if and only if the intuitionistic anti characteristic function $\lambda_Z = (\mu_{\lambda_Z}, \gamma_{\lambda_Z})$ of $Z = V \cap Q$ is an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2$.

Proof Let $Z = V \cap Q$ be a subring of ring $NR_1\times NR_2$ and let $v = (v_1, v_2)$, $r = (r_1, r_2) \in NR_1 \times NR_2$. If $v, r \in Z = V \cap Q$, then by properties of intuitionistic anti characteristic function $\mu_{\lambda_Z}(v) = 0 = \mu_{\lambda_Z}(r)$ and $\gamma_{\lambda_Z}(v) = 1 = \gamma_{\lambda_Z}(r)$. Since $v - r$ and $vr \in Z$. Then, $\mu_{\lambda_Z}(v - r) = 0 = 0 \diamond 0 = \mu_{\lambda_Z}(v) \diamond \mu_{\lambda_Z}(r)$, $\mu_{\lambda_Z}(vr) = 0 = 0 \diamond 0 = \mu_{\lambda_Z}(v) \diamond \mu_{\lambda_Z}(r)$

and $\gamma_{\lambda_Z}(v - r) = 1 = 1 * 1 = \gamma_{\lambda_Z}(v) * \gamma_{\lambda_Z}(r)$, $\gamma_{\lambda_Z}(vr) = 1 = 1 * 1 = \gamma_{\lambda_Z}(v) * \gamma_{\lambda_Z}(r)$.

Therefore,

$$\mu_{\lambda_Z}(v - r) \leq \mu_{\lambda_Z}(v) \diamond \mu_{\lambda_Z}(r),$$
$$\mu_{\lambda_Z}(vr) \leq \mu_{\lambda_Z}(v) \diamond \mu_{\lambda_Z}(r),$$
$$\gamma_{\lambda_Z}(v - r) \geq \gamma_{\lambda_Z}(v) * \gamma_{\lambda_Z}(r),$$
$$\gamma_{\lambda_Z}(vr) \geq \gamma_{\lambda_Z}(v) * \gamma_{\lambda_Z}(r).$$

Since, vr and $rv \in Z$, then $\mu_{\lambda_Z}(vr) = 0 = \mu_{\lambda_Z}(rv)$ and $\gamma_{\lambda_Z}(vr) = 1 = \gamma_{\lambda_Z}(rv)$ so $\mu_{\lambda_Z}(vr) = \mu_{\lambda_Z}(rv)$ and $\gamma_{\lambda_Z}(vr) = \gamma_{\lambda_Z}(rv)$. We also have when $v, r \notin Z$:

$$\mu_{\lambda_Z}(v - r) \leq \mu_{\lambda_Z}(v) \diamond \mu_{\lambda_Z}(r),$$
$$\mu_{\lambda_Z}(vr) \leq \mu_{\lambda_Z}(v) \diamond \mu_{\lambda_Z}(r),$$
$$\gamma_{\lambda_Z}(v - r) \geq \gamma_{\lambda_Z}(v) * \gamma_{\lambda_Z}(r),$$
$$\gamma_{\lambda_Z}(vr) \geq \gamma_{\lambda_Z}(v) * \gamma_{\lambda_Z}(r).$$

Also, $\mu_{\lambda_Z}(vr) = \mu_{\lambda_Z}(rv)$ and $\gamma_{\lambda_Z}(vr) = \gamma_{\lambda_Z}(rv)$.

Hence, the intuitionistic anti characteristic function $\lambda_Z = (\mu_{\lambda_Z}, \gamma_{\lambda_Z})$ of Z is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2$.

Conversely, assume that the intuitionistic anti characteristic function $\lambda_Z = (\mu_{\lambda_Z}, \gamma_{\lambda_Z})$ is an intuitionistic anti fuzzy normed normal subring. Let $v, r \in Z = V \cap Q$, then $\mu_{\lambda_Z}(v) = 0 = \mu_{\lambda_Z}(r)$ and $\gamma_{\lambda_Z}(v) = 1 = \gamma_{\lambda_Z}(r)$, hence:

$$\mu_{\lambda_Z}(v - r) \leq \mu_{\lambda_Z}(v) \diamond \mu_{\lambda_Z}(r) = 0 \diamond 0 = 0,$$

$$\mu_{\lambda_Z}(vr) \leq \mu_{\lambda_Z}(v) \diamond \mu_{\lambda_Z}(r) = 0 \diamond 0 = 0,$$

$$\gamma_{\lambda_Z}(v - r) \geq \gamma_{\lambda_Z}(v) * \gamma_{\lambda_Z}(r) = 1 * 1 = 1,$$

$$\gamma_{\lambda_Z}(vr) \geq \gamma_{\lambda_Z}(v) * \gamma_{\lambda_Z}(r) = 1 * 1 = 1.$$

Thus $\mu_{\lambda_Z}(v - r) = 0 = \mu_{\lambda_Z}(vr)$ and $\gamma_{\lambda_Z}(v - r) = 1 = \gamma_{\lambda_Z}(vr)$. This implies that $v - r$ and $vr \in Z$. Hence Z is a subring of ring $NR_1\times NR_2$.

Proposition 7 If the *IFS* $A\times B$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2$, then $\triangle\, A\times B = (\mu_{A\times B}, \mu^c_{A\times B})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2$.

Proof Let $A\times B$ be an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2$ and let $(v, r), (z, d) \in NR_1\times NR_2$. Then

$$\begin{aligned}
\mu^c_{A\times B}((v,r) - (z,d)) &= 1 - \mu_{A\times B}((v,r) - (z,d)) \\
&\geq 1 - (\mu_{A\times B}(v,r) \diamond \mu_{A\times B}(z,d)) \\
&= 1 - max\{\mu_{A\times B}(v,r), \mu_{A\times B}(z,d)\} \\
&= min\{1 - \mu_{A\times B}(v,r), 1 - \mu_{A\times B}(z,d)\} \\
&= min\{\mu^c_{A\times B}(v,r), \mu^c_{A\times B}(z,d)\} \\
&= \mu^c_{A\times B}(v,r) * \mu^c_{A\times B}(z,d)
\end{aligned}$$

and

$$\begin{aligned}
\mu^c_{A\times B}((v,r) \circ (z,d)) &= 1 - \mu_{A\times B}((v,r) \circ (z,d)) \\
&\geq 1 - (\mu_{A\times B}(v,r) \diamond \mu_{A\times B}(z,d)) \\
&= 1 - max\{\mu_{A\times B}(v,r), \mu_{A\times B}(z,d)\} \\
&= min\{1 - \mu_{A\times B}(v,r), 1 - \mu_{A\times B}(z,d)\} \\
&= min\{\mu^c_{A\times B}(v,r), \mu^c_{A\times B}(z,d)\} \\
&= \mu^c_{A\times B}(v,r) * \mu^c_{A\times B}(z,d).
\end{aligned}$$

Thus $\triangle\, A\times B = (\mu_{A\times B}, \mu^c_{A\times B})$ is an intuitionistic anti fuzzy normed subring $NR_1\times NR_2$.

$$\begin{aligned}
\mu^c_{A\times B}((v,r) \circ (z,d)) &= 1 - \mu_{A\times B}((v,r) \circ (z,d)) \\
&= 1 - \mu_{A\times B}((z,d) \circ (v,r)) \\
&= \mu^c_{A\times B}\big((z,d) \circ (v,r)\big).
\end{aligned}$$

Hence, $\triangle\, A\times B = (\mu_{A\times B}, \mu^c_{A\times B})$ is an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2$.

Proposition 8 If the *IFS* $A\times B$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2$, then $\Diamond\, A\times B = (\gamma^c_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2$.

Proof Similar to the proof of Proposition 7.

Corollary 1 An *IFS* $A \times B$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$ if and only if $\triangle\, A \times B = (\mu_{A\times B}, \mu^c_{A\times B})$ (resp.$\lozenge\, A \times B = (\gamma^c_{A\times B}, \gamma_{A\times B})$) is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$.

Theorem 3 An *IFS* $A \times B$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$ if and only if the fuzzy subsets $\mu_{A\times B}$ and $\gamma^c_{A\times B}$ are intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$.

Proof Let $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ be an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$. This implies that $\mu_{A\times B}$ is an intuitionistic anti fuzzy normed normal subring of $NR_1 \times NR_2$. We have to show that $\gamma^c_{A\times B}$ is also an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$. Let $(v,r),(z,d) \in NR_1 \times NR_2$. Then

$$\begin{aligned}\gamma^c_{A\times B}((v,r)-(z,d)) &= 1-\gamma_{A\times B}((v,r)-(z,d))\\ &\le 1-(\gamma_{A\times B}(v,r) * \gamma_{A\times B}(z,d))\\ &= 1-min\{\gamma_{A\times B}(v,r),\gamma_{A\times B}(z,d)\}\\ &= max\{1-\gamma_{A\times B}(v,r), 1-\gamma_{A\times B}(z,d)\}\\ &= max\{\gamma^c_{A\times B}(v,r),\gamma^c_{A\times B}(z,d)\}\\ &= \gamma^c_{A\times B}(v,r) \diamond \gamma^c_{A\times B}(z,d)\end{aligned}$$

and

$$\begin{aligned}\gamma^c_{A\times B}((v,r)\circ(z,d)) &= 1-\gamma_{A\times B}((v,r)-(z,d))\\ &\le 1-(\gamma_{A\times B}(v,r) * \gamma_{A\times B}(z,d))\\ &= 1-min\{\gamma_{A\times B}(v,r),\gamma_{A\times B}(z,d)\}\\ &= max\{1-\gamma_{A\times B}(v,r), 1-\gamma_{A\times B}(z,d)\}\\ &= max\{\gamma^c_{A\times B}(v,r),\gamma^c_{A\times B}(z,d)\}\\ &= \gamma^c_{A\times B}(v,r) \diamond \gamma^c_{A\times B}(z,d).\end{aligned}$$

Hence, $\gamma^c{}_{A\times B}$ is also an intuitionistic anti fuzzy normed subring of the ring $NR_1 \times NR_2$.

$$\begin{aligned}\gamma^c_{A\times B}((v,r)\circ(z,d)) &= 1-\gamma_{A\times B}((v,r)\circ(z,d))\\ &= 1-\gamma_{A\times B}((z,d)\circ(v,r))\\ &= \gamma^c_{A\times B}((z,d)\circ(v,r)).\end{aligned}$$

Hence, $\gamma^c_{A\times B}$ is an intuitionistic anti fuzzy normed normal subring of $NR_1 \times NR_2$.

Conversely, suppose that $\mu_{A\times B}$ and $\gamma^c_{A\times B}$ are intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$. We have to show that $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$. Then,

$$\begin{aligned}1-\gamma_{A\times B}((v,r)-(z,d)) &= \gamma^c_{A\times B}((v,r)-(z,d))\\ &\le \gamma^c_{A\times B}(v,r) \diamond \gamma^c_{A\times B}(z,d)\\ &= max\{\gamma^c_{A\times B}(v,r),\gamma^c_{A\times B}(z,d)\}\\ &= max\{1-\gamma_{A\times B}(v,r),1-\gamma_{A\times B}(z,d)\}\\ &= 1-min\{\gamma_{A\times B}(v,r),\gamma_{A\times B}(z,d)\}\\ &= 1-(\gamma_{A\times B}(v,r) * \gamma_{A\times B}(z,d))\end{aligned}$$

and

$$\begin{aligned}1-\gamma_{A\times B}((v,r)\circ(z,d)) &= \gamma^c_{A\times B}((v,r)\circ(z,d))\\ &\le \gamma^c_{A\times B}(v,r) \diamond \gamma^c_{A\times B}(z,d)\\ &= max\{\gamma^c_{A\times B}(v,r),\gamma^c_{A\times B}(z,d)\}\\ &= max\{1-\gamma_{A\times B}(v,r),1-\gamma_{A\times B}(z,d)\}\\ &= 1-min\{\gamma_{A\times B}(v,r),\gamma_{A\times B}(z,d)\}\\ &= 1-(\gamma_{A\times B}(v,r) * \gamma_{A\times B}(z,d)).\end{aligned}$$

Therefore, $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy normed subring of the ring $NR_1 \times NR_2$.

$$\begin{aligned}1-\gamma_{A\times B}((v,r)\circ(z,d)) &= \gamma^c_{A\times B}((v,r)\circ(z,d))\\ &= \gamma^c_{A\times B}((z,d)\circ(v,r))\\ &= 1-\gamma_{A\times B}((z,d)\circ(v,r)).\end{aligned}$$

Therefore, $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$.

Theorem 4 An *IFS* $A \times B$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$ if and only if the fuzzy subsets $\mu^c_{A\times B}$ and $\gamma_{A\times B}$ are intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$.

Proof Let $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ be an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$. This implies that $\gamma_{A\times B}$ is an intuitionistic anti fuzzy normed normal subring of $NR_1 \times NR_2$. We need to show that $\mu^c_{A\times B}$ is also an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$. The proof is similar to the first part of Proposition 7.

Conversely, suppose that $\mu^c_{A\times B}$ and $\gamma_{A\times B}$ are intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$. We need to show that $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$.

Then,

$$\begin{aligned}1-\mu_{A\times B}((v,r)-(z,d)) &= \mu^c_{A\times B}((v,r)-(z,d))\\ &\ge \mu^c_{A\times B}(v,r) * \mu^c_{A\times B}(z,d)\\ &= min\{\mu^c_{A\times B}(v,r),\mu^c_{A\times B}(z,d)\}\\ &= min\{1-\mu_{A\times B}(v,r), 1-\mu_{A\times B}(z,d)\}\\ &= 1-max\{\mu_{A\times B}(v,r),\mu_{A\times B}(z,d)\}\\ &= 1-(\mu_{A\times B}(v,r) \diamond \mu_{A\times B}(z,d))\end{aligned}$$

and

$$\begin{aligned} 1-\mu_{A\times B}((v,r)\circ(z,d)) &= \mu^c_{A\times B}((v,r)\circ(z,d)) \\ &\geq \mu^c_{A\times B}(v,r) * \mu^c_{A\times B}(z,d) \\ &= min\{\mu^c_{A\times B}(v,r), \mu^c_{A\times B}(z,d)\} \\ &= min\{1-\mu_{A\times B}(v,r), 1-\mu_{A\times B}(z,d)\} \\ &= 1-max\{\mu_{A\times B}(v,r), \mu_{A\times B}(z,d)\} \\ &= 1-(\mu_{A\times B}(v,r) \diamond \mu_{A\times B}(z,d)). \end{aligned}$$

Therefore, $A\times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy normed subring of the ring $NR_1\times NR_2$.

$$\begin{aligned} 1-\mu_{A\times B}((v,r)\circ(z,d)) &= \mu^c_{A\times B}((v,r)\circ(z,d)) \\ &= \mu^c_{A\times B}((z,d)\circ(v,r)) \\ &= 1-\mu_{A\times B}((z,d)\circ(v,r)). \end{aligned}$$

Therefore, $A \times B = (\mu_{A\times B}, \gamma_{A\times B})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2$.

Conclusion

The conception of intuitionistic anti fuzzy normed normal subrings has been extended for intuitionistic fuzzy normed subrings. We characterized direct product for intuitionistic anti fuzzy normed normal subrings with respect to (t,s)-norms and deduced several results. Finally, we established the relation between intuitionistic anti characteristic function and intuitionistic anti fuzzy normed normal subrings and some of its properties were investigated.

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