

## The Effect of the Quantization of the Centrifugal Stretching on the Analysis of the Rotational Spectra of Even-Even Nuclei

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### ABSTRACT

An approach based on the idea that the spinning nucleus being stretched out along the symmetry axis under the influence of some of centrifugal force has been proposed. The stretching in this work is treated within the framework of quantum mechanics rather than classical mechanics which had been used by Diamond Stephens and Swiatecki. Our approach led to a new formula that describes the dependence of the moment of inertia on the angular momentum. This formula is applied for the calculation of rotational ground state bands of even-even nuclei in the atomic mass range  $150 < A < 190$  and that having energy ratios in the range between  $2.9 \leq E_4/E_2 \leq 3.33$ . The results show an overall agreement with the experimental data up to high level energies. There are a small and systematic deviation appears at  $I > 12$ . This deviation increases with the increasing in  $I$  and also differs from one nucleus to another.

Keywords: Bands; energy level; model; properties; rotational; stretching

### ABSTRAK

Pendekatan berdasarkan idea bahawa nukleus berputar yang meregang di sepanjang paksi simetri di bawah pengaruh beberapa daya pengempar telah diusulkan. Peregangan dalam kertas ini dirawat dalam kerangka mekanik kuantum dan bukan mekanik klasik yang telah digunakan oleh Diamond Stephens dan Swiatecki. Pendekatan kami membawa kepada formula baharu yang menunjukkan pergantungan momen inersia pada momentum sudut. Formula ini digunakan untuk pengiraan jalur keadaan asas putaran nukleus genap-genap dalam julat jisim atom  $150 < A < 190$  dan mempunyai nisbah tenaga dalam julat antara  $2.9 \leq E_4/E_2 \leq 3.33$ . Hasilnya menunjukkan kesepakatan keseluruhan dengan data uji kaji hingga tenaga tahap tinggi. Terdapat penyimpangan kecil dan sistematik yang muncul pada  $I > 12$ . Penyimpangan ini meningkat dengan peningkatan  $I$  dan juga berbeza daripada satu nukleus ke nukleus yang lain.

Kata kunci: Jalur; model; putaran; regangan; tahap tenaga

### INTRODUCTION

Bohr, in his model (1976, 1952, 1951) of describing the nucleus in terms of the coupled particle motion and surface oscillations predicted, in the special case of the strong-coupling, the low-lying states of nuclear spectrum is rotational with spin sequence  $I = 0, 2, 4, \dots$  even parity and with level energies follow the simple formula

$$E(I) = \frac{\hbar^2}{2J} [I(I+1)] \quad (1)$$

where  $J$  is the moment of inertia. However, it is found later that the observed energy spacing of the states of the ground band increase less rapidly than (1) predicted when  $J$  is constant (Sorensen 1973). Bohr and Mottelson (1953) suggested that this decrease in the energy spacing

may be understood within the context of the coupling between rotational and vibrational modes of motion which contributes a term of the form  $-B[I(I+1)]^2$  and hence (1) becomes

$$E(I) = A[I(I+1)] - B[I(I+1)]^2 \quad (2)$$

The two parameters  $A$  and  $B$  in this equation are related to the moment of inertia and to the rotation-vibration coupling coefficient, respectively. However, such correction was found to be insufficient to represent the experimental spectra obtained with the states with higher values of  $I$  or even for low lying states of transitional nuclei (Preston & Bhaduri 1975). One can generalize (2) to an infinite power series form (Sood 1967)

$$E(I) = A \cdot I(I+1) - B[I(I+1)]^2 + C[I(I+1)]^3 - D[I(I+1)]^4 + \dots, \quad (3)$$

where  $A, B, C, D, \dots$  are parameters which can be determined by fitting this equation with the experimental results. The ground rotational bands of actinide and of rare-earth even-even nuclei were analyzed using the first four terms of (3) (Xu et al. 1989), where the parameters  $A, B, C$ , and  $D$  were obtained by the least-squares fitting. Although the success of this approach gave a good agreement with experiment for the data available up a very high  $I$  is impressive, it should be noted that this approach do not explain the physical meaning of the parameters  $C$  and  $D$ .

Harris (1964) suggested that  $E$  can be expanded in powers of  $\omega^2$  instead of  $I(I+1)$ . It turned out that the expansion in  $\omega^2$  converges more rapidly than that of  $I(I+1)$ . This feature could be exploited to find relations between higher parameters in (3). Particularly, considering only the first two terms in the expansion of Harris imply the relations (Bohr & Mottelson 1998)

$$\frac{C}{A} = 4\left(\frac{B}{A}\right)^2, \quad \frac{D}{A} = 24\left(\frac{B}{A}\right)^3.$$

These equations were compared with the values of  $A, B, C$ , and  $D$  obtained using least square fitting technique for several nuclei in the rare-earth region, it is seen that these equations are obeyed in most cases to around 70% (Bohr & Mottelson 1998), it can be seen Table 1 and Figures 1 and 2 (Xu et al. 1989). In any way, the problem of writing the coefficient of higher terms in

(3) in terms of first two basic parameters  $A$  and  $B$  was discussed by Gupta (1969, 1967) and later Trainor and Gupta (1971) introduces a model with a rotational-vibrational interaction that relates all parameters  $C, D, E, \dots$  to the first two basic parameters  $A$  and  $B$ . This model is known as non-rigid rotator model and the energy states are expressed as a power series in terms of  $B/A$  and the weight factor  $I(I+1)$  as:

$$E(I) = A \cdot I(I+1) \left[ 1 - \frac{B}{A} I(I+1) + 3 \left( \frac{B}{A} I(I+1) \right)^2 - \dots \right], \quad (4)$$

It has been shown that the first three terms and even the first four terms of (4) are capable of solving the problem only within a very short range  $3.25 < R < 3.33$  (Figure 1).

A detailed analysis by Sood (1968a, 1968b, 1967) and Volkov (1971) showed that the use of any truncated series in  $I(I+1)$  in (4) is not sufficient to describe the experimental spectrum. Instead, he suggested that the phenomenological sum an infinite term in  $I(I+1)$  of that equation can give the following compact expression for the energy  $E(I)$

$$E(I) = A(I) \cdot I(I+1), \quad (5)$$

where  $A(I) = A \left[ 1 - \frac{(B/A)I(I+1)}{1 + N(B/A)I(I+1)} \right]$ , which could be consistent with that derived by Bohr for describing the experimental spectrum with the provision that  $A(I)$  in (5) is variable and it is a decreasing function of  $I$  ( $J$  is increasing as function of  $I$ ). Although this two-parameter model by Sood (1967) fits with high accuracy the experimental spectrum of rare-earth nuclei including the  $Os$  isotopes and  $N = 90$  nuclei, there are many questions about the parameter  $N$  has been left open such as; what is the physical quantity that the parameter  $N$  in (5) represents? Or what are the factors that  $N$  depends on? The answer to these questions is very important to specify the factors which control the increase of  $J$  with the increase in  $I$ .

An alternative approach to interpret such decreasing in the energy space was introduced first by Morinaga (1966) and later by Diamond et al. (1964). Morinaga was the first to suggest that the decrease in energy spacing is due to the increase in the moment of inertia  $J$ . Diamond et al. (1964) attributed this increase in  $J$  to some sort of centrifugal stretching. The authors assumed

the suggestion of Bohr that the energy of an axially symmetric nucleus should include, in addition to the rotational energy, a potential energy

$$E(I) = \frac{\hbar^2}{2J} [I(I+1)] + \frac{1}{2} C (\beta_I - \beta_0)^2, \quad (6)$$

where  $J$  is moment of inertia of the nuclei and  $C$  is the stiffness of the nucleus. The authors assume the hydrodynamical formula of  $J$  (i.e.,  $J \propto \beta_I$ ). Where  $\beta_I$  in (6) is the value of  $\beta$  which satisfied the condition

$$\frac{\partial E(I)}{\partial \beta_I} = 0, \quad (7)$$

so that the total energy of that state is minimized. This approach gives good fit only for strongly deformed nuclei. Bands outside this region cannot be fitted with reasonable accuracy.

Holmberg and Lipas (1968) conducted a study based on the idea of Diamond et al. (1964), in which they derived a new two parameters formula for the level energies in the ground state bands called ab formula

$$E = a \left[ \sqrt{1 + b \cdot I(I+1)} - 1 \right], \text{ where } a, b \text{ are constants.}$$

This formula was examined by many authors such as Wu et al. (1992) who used this formula to determine the spins of 22 super deformed bands observed in the  $A \sim 190$  region. The agreement between the calculated and observed transition energies was incredibly well when a correct spin assignment is made. In fact, this result is expected in advance as we mentioned earlier, since the idea of the minimal energy of Diamond works well in the case of the super deformed nuclei. Authors Hu and Zeng (1997) made a comparison of the Harris and ab expressions for the description of nuclear normally deformed and super-deformed bands. They show that in normally deformed nuclei, there exist an obvious and systematic deviation of the Harris formula from the experiment. In contrast, the prediction of the ab formula is very close to the experiment, and maybe conveniently used for the description of both normally deformed and super-deformed bands.

A modification of the ab formula was made by Huang et al. (1989) who take into consideration small axial symmetry and vibrational effect. This approach led to a derivation of a three-parameter formula. The authors claimed that the third term  $C$ , describes the effect of anharmonicity but there is no evidence to support this

claim. However, this obtained modified formula gives an excellent agreement with experiment for all actinide and rare-earth deformed nuclei up to a very high spin.

Mariscotti et al. (1969) claimed that the responsibility for the failure of Diamond's model to describe the spectrum of nuclei outside the deformability region is that the increase in beta is not large enough to explain the deviation from rule and hence, they assumed that the moment of inertia  $J$  depends, in addition to the deformation parameter  $\beta$ , on pairing effect, they suggested  $\beta$  in (7) should be replaced by a general variable  $t$  which might represent not only the deformation parameter, but also all other microscopic features like the effective pairing. The dependence of  $J$  on  $t$  can be expressed as  $J = \text{const.} \cdot t^n$ ,  $n$  being integer. Since the best fit for all ground state bands ranging  $2.34 \leq R \leq 3.33$  were obtained  $n = 1$ , then  $J$  itself can be considered as a general variable and the equation of the total energy for the ground state bands, (7), takes the form,

$$E(I) = \frac{\hbar^2}{2J} [I(I+1)] + \frac{1}{2} C (J_I - J_0)^2, \quad (8)$$

The authors assumed that the minimal energy principle

$$\frac{\partial E(I)}{\partial J} = 0, \quad (9)$$

must be satisfied for each state of  $I$ . The model of the Mariscotti et al. (1969) is known as the variable moment of inertia (VMI) model. The limit of validity of this model was taken for  $J_0 > 0$  which is equivalent to  $> 2.34$ . An extension for this model to permit negative values of  $J_0$  were proposed by Scharff-Goldhaber and Goldhaber (1970). Another extension was made by Toki and Faessler (1976) to include asymmetric deformed nuclei.

Another work in which the Coriolis mixture of low-lying energy state bands were considered to analyze the deviation from the adiabatic condition i.e.,  $I(I+1)$  rule of even-even deformed nuclei was proposed (Okhunov et al. 2015; Usmanov et al. 2021, 2019, 2018, & 2010). In that model, the moment inertia  $J_0$  and  $J_1$  by Harris parametrization for the energy and angular momentum were calculated and the obtained values were used in the calculation of the energy levels of the ground state bands for several isotopes in rare-earth region. The results obtained show a very good agreement with experimental data.

Recently, the effects of possible non collective pairs in even-even  $^{124-128}Sn$  are studied in the nucleon-pair shell

model (He et al. 2020), and also the effective moment of inertia of even-even rotating nuclei have been studied (El Sheikh et al. 2020) by expanding the Bohr's equation into account the second and third terms of this expansion in addition to first one. The modified form of Bohr's relationship, is verified on axially symmetric nuclei of atomic mass ranging between 150 and 190 as well as on a number of triaxially symmetric nuclei.

In this work, we follow the suggestion of Diamond et al. (1964), that the spinning nucleus being stretched along its symmetry axis with the exception that the stretching is treated in our work according to quantum mechanics rather than classical mechanics which was already used in their work which leads to a new formula that describe the dependence of the moment of inertia on the angular momentum. This formula has been applied to calculate the level energies of the ground state bands of a large number of nuclei in the strongly deformed and in the transitional regions that are in the atomic range  $150 < A < 190$  and having energy ratios  $2.9 < R < 3.33$ . In the case of strongly deformed nuclei this formula shows an excellent agreement with experiments up to very high energy levels (Figure 4), for moderate deformed and transitional nuclei there are small and systematic deviations that appear in the form of dispersion around the theoretical curve. In our work, we only consider the effect of the stretching we compare our results with, other approaches who consider in addition to the stretching other factors like pairing and Coriolis forces (Mariscotti et al. 1969; Okhunov et al. 2015) in order to find how much these factors, contribute to the moment of inertia and, accordingly, to the values of level energies of the nuclei considered in this work.

A brief description of the formulation of this formula will be displayed in the next section. Application of this formula to the ground state bands of an even-even nuclei whose energy ratio not less than 2.9 is given subsequently. Lastly, in the final section, the important conclusions will be summarized.

#### FORMALISM

Following the suggestion of Diamond et al. (1964), the increase in the moment of inertia can be interpreted on the basis of the idea that spinning nucleus exhibits some sort of centrifugal stretching along the symmetric axis, the very simple classical form of the moment of inertia of the mass element  $dm$  is

$$dJ = r^2 dm, \quad (10)$$

where  $r$  is the effective radius of rotation. According to the hydrodynamical model, very little of the nuclear matter is actually taking part in the effective rotational motion or, in other words, the rotational motion can be pictured as a motion of wave around the nuclear surface, so

$$dm = \frac{M}{A} R^2 d\Omega = \frac{M}{4\pi} d\Omega, \quad (11)$$

where  $A \approx 4\pi R^2$  is the total area of the surface,  $R^2 d\Omega \approx R^2 \sin \theta d\theta d\phi$  is the surface element,  $R$  is the distance of the mass element  $dm$  from the center of the nucleus. Where  $r$  in (10) can be written as  $r_0 + \Delta r$ , where  $\Delta r$  is the stretched in the nucleus due to rotation. It follows

$$dJ = (r_0 + \Delta r)^2 \frac{M}{4\pi} d\Omega = r_0^2 \left(1 + \frac{\Delta r}{r_0}\right)^2 \frac{M}{4\pi} d\Omega \quad (12)$$

We assume that the nucleus has symmetry axis which is perpendicular to its rotational axis then the stretched can be expanded in body fixed-space in terms of a complete set of spherical harmonics (Eisenberg & Greiner 1987) as  $\frac{\Delta r}{r_0} = \sum_{\lambda\mu} a_{\lambda\mu}^* Y_{\lambda\mu}(\theta, \phi)$ , where  $a_{\lambda\mu}^*$  is the deformation parameter in body-fixed coordinate. Putting this definition in (12) we get

$$\begin{aligned} dJ &= r_0^2 \left(1 + \sum_{\lambda\mu} a_{\lambda\mu}^* Y_{\lambda\mu}(\theta, \phi)\right)^2 \frac{M}{4\pi} d\Omega \\ &= r_0^2 \left(1 + 2 \sum_{\lambda\mu} a_{\lambda\mu}^* Y_{\lambda\mu}(\theta, \phi) + \sum_{\lambda\lambda', \mu\mu'} a_{\lambda\mu}^* a_{\lambda'\mu'}^* Y_{\lambda\mu} Y_{\lambda'\mu'}\right) \frac{M}{4\pi} d\Omega \end{aligned} \quad (13)$$

Integrating both sides of (13) over the whole surface of the nucleus, one can find

$$\begin{aligned} J &= r_0^2 \left( \int d\Omega + 2 \sum_{\lambda\mu} a_{\lambda\mu}^* \int Y_{\lambda\mu}(\theta, \phi) d\Omega + \right. \\ &\quad \left. \sum_{\lambda\lambda', \mu\mu'} a_{\lambda\mu}^* a_{\lambda'\mu'}^* \int Y_{\lambda\mu} Y_{\lambda'\mu'} d\Omega \right) \frac{M}{4\pi}. \end{aligned} \quad (14)$$

The integration  $\int d\Omega = 4\pi$ , refer to the integration over solid angle, it is well known that the integration  $\int Y_{\lambda\mu}(\theta, \phi) d\Omega$  which is the second term in the above equation is zero for all  $\lambda > 0$  and  $\int Y_{\lambda\mu} Y_{\lambda'\mu'} d\Omega = (-1)^\mu \delta_{\lambda\lambda'} \delta_{\mu-\mu'}$ . The quantity  $(-1)^\mu a_{\lambda'\mu'}$  equals to  $a_{\lambda'-\mu'}$ . Putting all these requirements in (14) we get

$$J = J_0 \left(1 + \frac{1}{4\pi} \sum_{\lambda\mu} (-1)^\mu a_{\lambda, -\mu} a_{\lambda, \mu}\right) \quad (15)$$

where  $a_{\lambda,\mu}$  is the deformation parameter in body-fixed coordinates. It is found that even-even nuclei can be accurately described in terms of a deformation of order  $\lambda = 2$ . In (15), the suffix index  $\mu$  which represents the orientation of the nucleus in space-fixed coordinates runs from  $-\lambda$  to  $\lambda$ . In our case  $\mu$  runs from  $-2$  to  $2$ . We drop the subscript  $\lambda = 2$  from all deformation parameters henceforth

$$J = J_0 \left( 1 + \frac{1}{4\pi} \sum_{\mu} (-1)^{\mu} a_{-\mu} a_{\mu} \right) \quad (16)$$

$$= J_0 \left\{ 1 + \frac{1}{4\pi 2C} \sum_{\mu} (-1)^{\mu} \left[ \zeta_{-\mu} + (-1)^{\mu} \zeta_{\mu}^* \right] \left[ \zeta_{\mu} + (-1)^{\mu} \zeta_{-\mu}^* \right] \right\}$$

Where  $\alpha_{\mu}$  can be written by using the identity  $\alpha_{\mu} = \sqrt{\frac{\hbar\omega}{2C}} (\zeta_{\mu} + (-1)^{\mu} \zeta_{-\mu}^*)$ . The quantity  $\left[ \zeta_{-\mu} + (-1)^{\mu} \zeta_{\mu}^* \right] \left[ \zeta_{\mu} + (-1)^{\mu} \zeta_{-\mu}^* \right]$  can be treated as follows

$$\left[ \zeta_{-\mu} + (-1)^{\mu} \zeta_{\mu}^* \right] \left[ \zeta_{\mu} + (-1)^{\mu} \zeta_{-\mu}^* \right] = \quad (17)$$

$$\zeta_{-\mu} \zeta_{\mu} + \zeta_{\mu}^* \zeta_{-\mu}^* + (-1)^{\mu} (\zeta_{-\mu} \zeta_{-\mu}^* + \zeta_{\mu}^* \zeta_{\mu})$$

Each of the first two terms in (17) includes two operators  $\zeta_{\mu}$  and  $\zeta_{-\mu}$  the first acts on the state  $\mu$  and the second on a different state  $-\mu$  but according to the selection rule which connects the ground state band to the one-phonon state, where we have only one state either  $\mu$  or  $-\mu$  these two terms should vanish. Using the commutation relation  $[\zeta_{-\mu}^* \zeta_{-\mu}] = 1$ , which leads to  $\zeta_{-\mu} \zeta_{-\mu}^* = 1 + \zeta_{-\mu}^* \zeta_{-\mu}$  the term  $(-1)^{\mu} (\zeta_{-\mu} \zeta_{-\mu}^* + \zeta_{\mu}^* \zeta_{\mu})$  in (17) becomes  $(-1)^{\mu} (1 + \zeta_{-\mu}^* \zeta_{-\mu} + \zeta_{\mu}^* \zeta_{\mu})$ . Therefore, (17) becomes

$$J_I = J_0 \left[ 1 + \frac{1}{4\pi} \frac{\hbar\omega}{2C} \left( \sum_{\mu} (1 + \zeta_{-\mu}^* \zeta_{-\mu} + \zeta_{\mu}^* \zeta_{\mu}) \right) \right] \quad (18)$$

Since the summation in (18) runs from  $-\mu$  to  $\mu$ , hence the summation over  $\zeta_{-\mu}^* \zeta_{-\mu}$  and  $\zeta_{\mu}^* \zeta_{\mu}$  are identical, then the (18) can be written in the form

$$J_I = J_0 \left[ 1 + \frac{1}{4\pi} \frac{\hbar\omega}{2C} \left( \sum_{\mu} (1 + 2\zeta_{\mu}^* \zeta_{\mu}) \right) \right] \quad (19)$$

$$= J_0 \left[ 1 + \frac{1}{4\pi} \frac{\hbar\omega}{2C} \left( \sum_{\mu} (1 + 2\hat{n}_{\mu}) \right) \right],$$

where  $\hat{n}_{\mu} = \zeta_{\mu}^* \zeta_{\mu}$  is the number operator. In the case of the ground state band the number operator is zero.

Replacing  $\hbar\omega$  by  $\frac{\hbar^2 I(I+1)}{2J_I}$  in (18) is reduced to

$$J_I = J_0 \left[ 1 + \frac{5}{4\pi} \frac{\hbar^2 I(I+1)}{4J_0 C} \right] = J_0 \left[ 1 + \frac{5}{4\pi} \frac{\hbar^2 I(I+1)}{4J_0 C \left[ 1 + \frac{5}{4\pi} \frac{\hbar^2 I(I+1)}{4J_0 C} \right]} \right]$$

$$\cong J_0 \left[ 1 + \frac{5}{4\pi} \frac{\hbar^2 I(I+1)}{4J_0 C + \frac{5}{4\pi} \hbar^2 I(I+1)} \right] \quad (20)$$

It is clear that (20) is recursion relation of  $J_I$  (i.e.  $J_I$  are defined in terms of itself). It represents the moment of inertia at angular momentum  $I$  approximated to the second order. The factor 5 in (20) arises because the summation over  $\mu$  runs from  $-2$  to  $2$  through  $\mu = 0$  as mentioned above. Finally, the energy levels in the ground state bands are casted into the following form,

$$E(I) = \frac{\hbar^2 I(I+1)}{2J_I} = \frac{\hbar^2 I(I+1)}{2J_0 \left[ 1 + \frac{5}{4\pi} \frac{\hbar^2 I(I+1)}{4J_0 C + \frac{5}{4\pi} \hbar^2 I(I+1)} \right]} \quad (21)$$

$$J \cong \frac{AI(I+1)}{\left[ 1 + \frac{BI(I+1)}{1+BI(I+1)} \right]}$$

where  $A = \frac{\hbar^2}{2J_0}$  and  $B = \frac{5}{4\pi} \frac{\hbar^2}{4J_0 C}$ . We will call the (21) quantized  $-\beta$  stretching equation. These parameters embed in them the intrinsic moment of inertia  $J_0$  and stiffness  $C$  of the nucleus. They are constant for a particular nucleus, but their values differ from one nucleus to another. In the present model, they are adjustable parameters and evaluated by fitting (21) against experimentally measured energy levels.

## RESULTS AND DISCUSSION

The simple expression (21), has been used to evaluate the level energies up to spin  $I = 20$ . The parameters  $A$  and  $B$  have been determined by means of the least square fitting method involving the first three experimentally measured energy levels (i.e.,  $I = 2, 4, 6$ ) in the ground state band. Our discussion includes all even-even isotopes  ${}_{62}\text{Sm}$ ,  ${}_{64}\text{Gd}$ ,  ${}_{66}\text{Dy}$ ,  ${}_{68}\text{Er}$ ,  ${}_{70}\text{Yb}$ ,  ${}_{72}\text{Hf}$ ,  ${}_{74}\text{W}$  and  ${}_{76}\text{Os}$  with the neutron numbers ranging from 90 to 114, which are found in the atomic mass range between 150 and 190 and having the energy ratio

$2,9 < E_4/E_2 < 3,33$ . Our results for the energy levels are data and the data obtained by the VMI model. presented in Table 1 which also includes the experimental

TABLE 1. The energy levels  $E(I)$  (keV) of the ground state bands of even-even isotopes of *Sm*, *Gd*, *Dy*, *Er*, *Yb*, *Hf* and *W*. (Exp. (NuDat 2.8.), *VMI* (Mariscotti et al. 1969), *CW* is current work)

Nuclei	$A$	$B$	Results	$I$							
				2	4	6	8	10	12	14	16
$^{152}\text{Sm}$	21.50	0.0105	Exp.	121.8	366.5	706.9	1125.4	1609.3	2148.8	2736.2	3365
			VMI	121.0	369.9	712.3	1127.3	1601.8	2127.2	2697.2	3307
			CW	121.8	366.5	691.6	1082.5	1540.2	2069.8	2675.4	3360.4
$^{154}\text{Sm}$	13.77	0.0016	Exp.	82.3	266.8	544.1	902.8	1333.0	1825.9	2373.0	2968.2
			VMI	81.5	267.7	550.4	920.3	1368.3	1886.5	2468.2	3107.7
			CW	81.8	267.0	544.0	898.7	1317.9	1791.7	2311.0	2875.5
$^{154}\text{Gd}$	20.88	0.0068	Exp.	123.1	371.0	717.7	1144.4	1637.1	2184.7	2777.3	3404.5
			VMI	122.0	374.4	722.8	1146.0	1630.7	2167.8	2750.9	3375.1
			CW	120.6	372.8	717.2	1130.7	1607.4	2149.0	2759.4	3442.2
$^{156}\text{Gd}$	14.94	0.0019	Exp.	89.0	288.2	584.7	965.1	1416.1	1924.5	2475.8	3059.5
			VMI	88.8	288.4	585.0	965.2	1417.9	1934.2	2506.9	3130.7
			CW	88.7	288.4	584.7	960.9	1402.5	1898.9	2444	3034.6
$^{158}\text{Gd}$	13.31	0.0009	Exp.	79.5	261.5	539.0	904.1	1349.5	1865.0		
			VMI	79.6	261.4	537.8	899.5	1338.0	1845.4	2415.3	3042.1
			CW	79.4	261.5	539.0	902.2	1340.5	1843.9	2404.0	3014.1
$^{160}\text{Gd}$	12.59	0.0007	Exp.	75.3	248.5	514.8	867.9	1300.7	1806.3	2377.3	3008.1
			VMI	75.1	247.5	511.5	860.0	1285.7	1781.8	2342.2	2961.7
			CW	75.2	248.5	514.8	866.9	1296.6	1795.5	2355.6	2970.2
$^{156}\text{Dy}$	23.72	0.0099	Exp.	137.8	404.2	770.4	1215.6	1725.0	2285.9	2887.8	3523.3
			VMI	137.1	407.3	769.3	1201.3	1690.3	2228.1	2808.5	3427.1
			CW	134.7	409.9	769.5	1205.0	1713.8	2301.1	2971.6	3729.5
$^{158}\text{Dy}$	16.52	0.0022	Exp.	98.8	317.1	637.7	1043.9	1520.0	2048.8	2612.2	3190.3
			VMI	98.8	317.0	635.3	1036.9	1509.2	2042.7	2630.3	3266.4
			CW	98.3	317.4	637.6	1038.5	1504.2	2025.1	2596.9	3255.7
$^{160}\text{Dy}$	14.56	0.0013	Exp.	86.8	283.8	581.1	966.9	1428.0	1950.5	2513.8	3089.8
			VMI	86.7	284.0	582.6	971.7	1441.5	1983.4	2590.2	3256.0
			CW	86.7	283.9	581.1	965.0	1422.3	1941.8	2515.0	3136.4
$^{162}\text{Dy}$	13.51	0.0009	Exp.	80.7	265.7	548.5	921.3	1375.1	1901.1	2491.7	3138.6
			VMI	80.9	266.2	549.2	921.5	1374.8	1901.6	2495.3	3150.2
			CW	81.7	265.7	548.5	919.7	1369.2	1887.1	2464.6	3094.9

$^{164}\text{Dy}$	12.29	0.0007	Exp.	73.4	242.2	501.3	843.7	1261.3	1745.9	2290.6	2887.1
			VMI	73.5	242.1	500.1	840.3	1255.4	1738.7	2284.2	2886.8
			CW	73.4	242.2	501.3	843.3	1260.0	1742.6	2283.9	2876.4
$^{160}\text{Er}$	21.24	0.0047	Exp.	125.8	389.9	765.7	1229.3	1761.1	2340.4	2932.7	3466.5
			VMI	125.8	392.5	767.1	1227.4	1758.4	2350.0	2994.9	3687.3
			CW	124.0	391.0	765.4	1219.8	1741.3	2326.7	2977.9	3698.2
$^{162}\text{Er}$	17.15	0.0021	Exp.	102.0	329.6	666.7	1096.7	1602.8	2165.1	2745.7	3292.4
			VMI	101.0	327.1	661.7	1089.1	1596.4	2173.5	2812.7	3507.7
			CW	101.6	329.8	666.7	1092.3	1590.1	2148.5	2761.3	3425.9
$^{164}\text{Er}$	15.32	0.0012	Exp.	91.4	299.4	614.4	1024.6	1518.1	2082.8	2702.6	3411.2
			VMI	90.9	297.6	610.0	1016.7	1507.1	2072.3	2704.7	3398.2
			CW	91.3	299.5	614.4	1022.9	1511.5	2068.4	2684.2	3352.6
$^{166}\text{Er}$	13.51	0.0010	Exp.	80.6	265.0	545.5	911.2	1349.5	1846.5	2389.3	2967.3
			VMI	80.6	264.8	544.6	910.6	1354.2	1867.3	2443.5	3077.0
			CW	80.6	265.0	545.5	911.7	1352.5	1857.8	2418.7	3029.1
$^{168}\text{Er}$	13.34	0.0005	Exp.	79.8	264.1	548.7	928.3	1396.8	1947.3	2571.9	3259.5
			VMI	79.8	264.1	548.9	928.9	1398	1950.2	2579.6	3281.1
			CW	79.8	264.1	548.7	927.5	1393.1	1937.7	2553.6	3233.4
$^{164}\text{Yb}$	20.84	0.0043	Exp.	123.3	385.6	760.1	1223.1	1753.4	2329.9	2899.2	3695.6
			VMI	122.0	386.4	756.9	1215.2	1737.8	2323.8	2972.7	3688.4
			CW	122.4	384.8	757.0	1217.2	1750.5	2346.4	2997.5	3697.9
$^{166}\text{Yb}$	17.21	0.0021	Exp.	102.4	330.5	668.0	1098.3	1605.9	2176.0	2779.5	3490.1
			VMI	102.0	330.7	668.0	1093.6	1590.6	2188.0	2759.0	3423.2
			CW	101.9	329.6	666.4	1096.1	1605.8	2185.3	2826.9	3524.2
$^{168}\text{Yb}$	14.73	0.0015	Exp.	87.7	286.6	585.3	970.0	1425.5	1936.0	2488.5	3073.1
			VMI	87.7	286.6	585.3	969.4	1425.0	1940.9	2509.1	3124.7
			CW	87.0	248.3	581.2	966.1	1428.7	1960.1	2553.3	3202.5
$^{170}\text{Yb}$	14.09	0.0008	Exp.	84.3	277.4	573.3	963.3	1437.5	1983.4	2580.4	3195.1
			VMI	84.1	277.5	573.3	962.4	1434.5	1979.5	2588.4	3253.6
			CW	84.2	277.4	572.9	962.4	1437.7	1990.9	2615.0	3304.9
$^{172}\text{Yb}$	13.16	0.0006	Exp.	78.7	260.3	540.0	912.1	1370.1	1907.5	2518.7	3198.4
			VMI	78.7	260.3	540.0	911.1	1365.7	1895.5	2492.5	3149.4
			CW	78.9	260.3	538.6	906.6	1357	1882.9	2477.8	3136.4
$^{174}\text{Yb}$	12.78	0.0005	Exp.	76.5	253.1	526.0	889.9	1336.0	1861.0	2457.0	3117.0
			VMI	76.3	252.9	526.3	891.9	1344.4	1878.3	2488.4	3169.8
			CW	76.5	253.1	526.0	889.3	1336.3	1859.4	2451.1	3104.8

			Exp.	82.1	271.9	564.5	953.9	1431	1984.6	2602.0	3270
$^{176}\text{Yb}$	13.74	0.0005	VMI	82.2	271.9	564.5	953.5	1431.1	1988.8	2618.6	3312.9
			CW	82.0	271.0	561.9	948.2	1422.9	1979.1	2610.6	3311.6
			Exp.	124.1	385.9	757.3	1213.7	1736.1	2306.1	2857.5	3310.4
$^{168}\text{Hf}$	21.01	0.0047	VMI	122.7	386.8	757.1	1206.5	1722.2	2301.1	2945.3	3657.7
			CW	123.8	386.2	755.0	1208.0	1730.7	2313	2947.8	3629.3
			Exp.	100.7	321.7	642.6	1043.0	1504.2	2015.9	2566.7	3151.1
$^{170}\text{Hf}$	17.01	0.0030	VMI	100.3	321.9	642.5	1040.1	1499.4	2012.6	2577.0	3193.1
			CW	101.0	320.7	636.7	1031.0	1491.0	2007.9	2574.5	3185.7
			Exp.	95.2	309.2	628.3	1037.5	1521.2	2064.7	2654.1	3277.2
$^{172}\text{Hf}$	16.00	0.0018	VMI	95.0	308.1	624.7	1030.2	1512.5	2062.2	2617.9	3335.6
			CW	95.0	309.3	628.3	1034.7	1512.6	2050.9	2642.2	3282.7
			Exp.	91.0	297.4	608.3	1009.6	1485.9	2020.5	2597.5	3208.9
$^{174}\text{Hf}$	15.29	0.0014	VMI	91.0	297.5	608.8	1013.1	1499.5	2059.0	2684.1	3368.7
			CW	90.9	297.5	608.3	1009.0	1485.0	2026.0	2622.0	3267.8
			Exp.	88.3	290.2	596.8	997.7	1481.1	2034.7	2646.6	3307.7
$^{176}\text{Hf}$	14.81	0.0011	VMI	88.1	289.6	296.2	998.3	1486.5	2052.3	2688.4	3388.7
			CW	88.3	290.2	596.8	996.4	1476.4	2025.8	2635.0	3297.3
			Exp.	93.2	306.6	632.2	1058.6	1570.3	2149.6	2776.6	3435.0
$^{178}\text{Hf}$	15.61	0.0009	VMI	93.2	306.7	632.1	1059.5	1579.1	2182.0	2860.7	3608.7
			CW	93.1	306.6	632.2	1058.6	1573.7	2165.6	2824.6	3542.7
			Exp.	93.3	308.6	640.8	1084	1631.0	2274.3	3005.4	3813.5
$^{180}\text{Hf}$	15.60	0.0005	VMI	93.3	308.7	641.3	1084.7	1631.6	2274.8	3007.4	3823.2
			CW	93.3	308.6	640.8	1082.4	1624.4	2257.4	2972.2	3760.2
			Exp.	113.0	356.4	706.4	1138.9	1637.5	2189.4	2785.2	3397.3
$^{174}\text{W}$	19.10	0.0038	VMI	112.2	354.9	701.8	1132.9	1634.4	2196.4	2811.6	3474.4
			CW	112.1	357.0	705.9	1133.7	1625.6	2175.7	2783.4	3450.9
			Exp.	108.3	348.2	699.4	1139.7	1648.5	2206.3	2802.6	3427.6
$^{176}\text{W}$	18.27	0.0026	VMI	109.1	348.7	696.4	1133.2	1645.4	2222.7	2857.4	3543.6
			CW	108.0	348.7	701.9	1148.2	1674.5	2274.3	2945.8	3689.6
			Exp.	105.9	342.7	694.2	1141.5	1665.4	2244.5	2858.7	3488.4
$^{178}\text{W}$	17.80	0.0020	VMI	105.4	341.4	690.7	1136.9	1666.5	2269.1	2936.5	3662.3
			CW	105.6	342.9	694.2	1139.1	1660.6	2245.8	2888.4	3584.9
			Exp.	103.6	337.4	688.5	1138.5	1664.2	2235.2	2822.9	3412.7
$^{180}\text{W}$	17.39	0.0016	VMI	103.1	335.7	683.5	1131.5	1667	2279.8	2961.4	3705.2
			CW	103.4	337.6	688.5	1138.2	1670.3	2271.7	2933.4	3650.1



			Exp.	100.1	329.4	680.4	1144.3	1712.0	2372.6	3112.9	3910.1
$^{182}\text{W}$	16.74	0.0008	VMI	100.0	329.2	678.9	1138.0	1698.7	2349.1	3081.9	3890.0
			CW	100.0	329.4	680.4	1141.5	1700.4	2344.9	3064.6	3849.8
			Exp.	111.2	364.1	748.3	1252.2	1860.8	2557.0	3319.9	4116.9
$^{184}\text{W}$	18.60	0.0011	VMI	111.1	364.4	747.8	1248.0	1852.3	2549.8	3331.3	4189.1
			CW	110.9	364.3	748.3	1247.7	1846.5	2530.3	3287.8	4110.5
			Exp.	122.6	396.5	809.3	1349.2	2002.4	2750.9	3562.4	
$^{186}\text{W}$	20.45	0.0016	VMI	122.4	399.8	817.2	1358.3	2008.3	2755.1	3588.5	4500.6
			CW	121.6	396.9	809.2	1337.3	1961.9	2667.6	3443.8	4284.6
			Exp.	126.9	400.3	794	1277.9	1812	2346.1	2840.7	3320.1
$^{182}\text{Os}$	21.41	0.0037	VMI	127.3	400.5	788.7	1268.9	1825.8	2448.5	3129	3861.2
			CW	125.8	401	793.9	1276.2	1831	2451.3	3136.2	3887.8
			Exp.	119.8	383.7	774.1	1274.8	1871.2	2547.6	3261.4	4046.5
$^{184}\text{Os}$	20.07	0.0023	VMI	119.4	385.0	775.4	1271.1	1856.8	2520.8	3254.2	4049.8
			CW	118.8	384.3	774.0	1263.6	1833.9	2472.4	3173.1	3934.1
			Exp.	137.2	434.1	868.9	1420.9	2068.0	2781.3	3557.7	
$^{186}\text{Os}$	23.00	0.0030	VMI	136.6	436.3	870.6	1415.8	2054.5	2774	3564.8	4419.5
			CW	135.4	435	869.0	1406.6	2028.6	2722.7	3486.5	4320
			Exp.	155.0	478.0	940.0	1514.8	2170.1	2856.3	3562.6	4236.5
$^{188}\text{Os}$	26.05	0.0047	VMI	154.3	481.4	941.5	1506.9	2159.6	2886.9	3679.8	4531.3
			CW	152.1	479.8	939.5	1497.7	2138.2	2857.1	3656.7	4540.8
			Exp.	186.7	547.9	1050.4	1666.8	2357.7			
$^{190}\text{Os}$	31.80	0.0089	VMI	185.3	554.6	1052.4	1648.5	2325.0	3069.8	3874.8	4733.4
			CW	181.5	552.1	1049.1	1645.1	2338.3	3134.4	4040.7	506.6

In Table 1, the data for VMI model are obtained from Mariscotti et al. (1969) and the experimental data from the decay data website <http://www.nndc.bnl.gov/nudat2/>. From the table we can see that the obtained results have a good agreement with the experimental data for most nuclei which is given in Table 1. Very few cases are found to display an error of 5% which arises because of the spread of the experimental points which increases rapidly with  $I$ . The results of this work and that of the VMI model are in most cases coincident. The present model has the advantage of being simpler and

clearer in the form of the energy levels than the VMI model.

Furthermore, in Table 2, the calculated values of the effective moment of inertia  $J_0$  for each nucleus considered are listed. The current work results are obtained using the formula  $J_0 = \frac{\hbar^2}{2A}$  where we have used the values of  $A$  from Table 1. The results of VMI are obtained from Table 2 (Mariscotti et al. 1969). Table 2 shows an excellent coincidence of the current results and that calculated according to VMI.

TABLE 2. The effective moment of inertia  $J_0$  calculated according to (20) in this work of even-even isotopes of *Sm*, *Gd*, *Dy*, *Er*, *Yb*, *Hf*, *W*, and *Os*. ( $J_0^{cw}$  is current work, and  $J_0^{VMI}$  is VMI (Mariscotti et al. 1969))

Nuclei	$J_0^{cw}$	$J_0^{VMI}$	Nuclei	$J_0^{cw}$	$J_0^{VMI}$
$^{152}Sm$	0.0233	0.0234	$^{174}Yb$	0.0391	0.0392
$^{154}Sm$	0.0363	0.0365	$^{176}Yb$	0.0364	0.0364
$^{152}Gd$	0.0239	0.0233	$^{168}Hf$	0.0238	0.0233
$^{156}Gd$	0.0335	0.0333	$^{170}Hf$	0.0294	0.0289
$^{158}Gd$	0.0376	0.0374	$^{172}Hf$	0.0313	0.0312
$^{160}Gd$	0.0397	0.0397	$^{152}Hf$	0.0327	0.0327
$^{156}Dy$	0.0211	0.0201	$^{152}Hf$	0.0338	0.0338
$^{158}Dy$	0.0303	0.0298	$^{178}Hf$	0.0320	0.0321
$^{160}Dy$	0.0343	0.0343	$^{180}Hf$	0.0321	0.0321
$^{162}Dy$	0.0370	0.0369	$^{174}W$	0.0262	0.0260
$^{164}Dy$	0.0407	0.0406	$^{176}W$	0.0274	0.0269
$^{160}Er$	0.0235	0.0229	$^{178}W$	0.0281	0.0280
$^{162}Er$	0.0292	0.0293	$^{180}W$	0.0288	0.0288
$^{164}Er$	0.0326	0.0327	$^{182}W$	0.0299	0.0298
$^{166}Er$	0.0370	0.0369	$^{184}W$	0.0269	0.0268
$^{168}Er$	0.0375	0.0375	$^{186}W$	0.0244	0.0243
$^{170}Er$	0.0381	0.0378	$^{182}Os$	0.0234	0.0228
$^{164}Yb$	0.0240	0.0237	$^{184}Os$	0.0249	0.0247
$^{166}Yb$	0.0291	0.0289	$^{186}Os$	0.0217	0.0215
$^{168}Yb$	0.0339	0.0342	$^{188}Os$	0.0192	0.0187
$^{170}Yb$	0.0355	0.0354	$^{190}Os$	0.0157	0.0150
$^{172}Yb$	0.0380	0.0379			

The results of the level energies of the ground state bands calculated using our formula (21) for  $^{70}Yb^{170-176}$ ,  $^{72}Hf^{170-180}$ ,  $^{74}W^{174-178}$  isotopes were compared with VMI model (Mariscotti et al. 1969) mentioned before and with the results based on the phenomenological

model which takes into account the Coriolis mixture of low-lying state bands (Okhunov et al. 2015). This comparison is shown in Figures 1 and 2 which includes also experimental data obtained from the decay data website <http://www.nndc.bnl.gov/nucdat2/>.

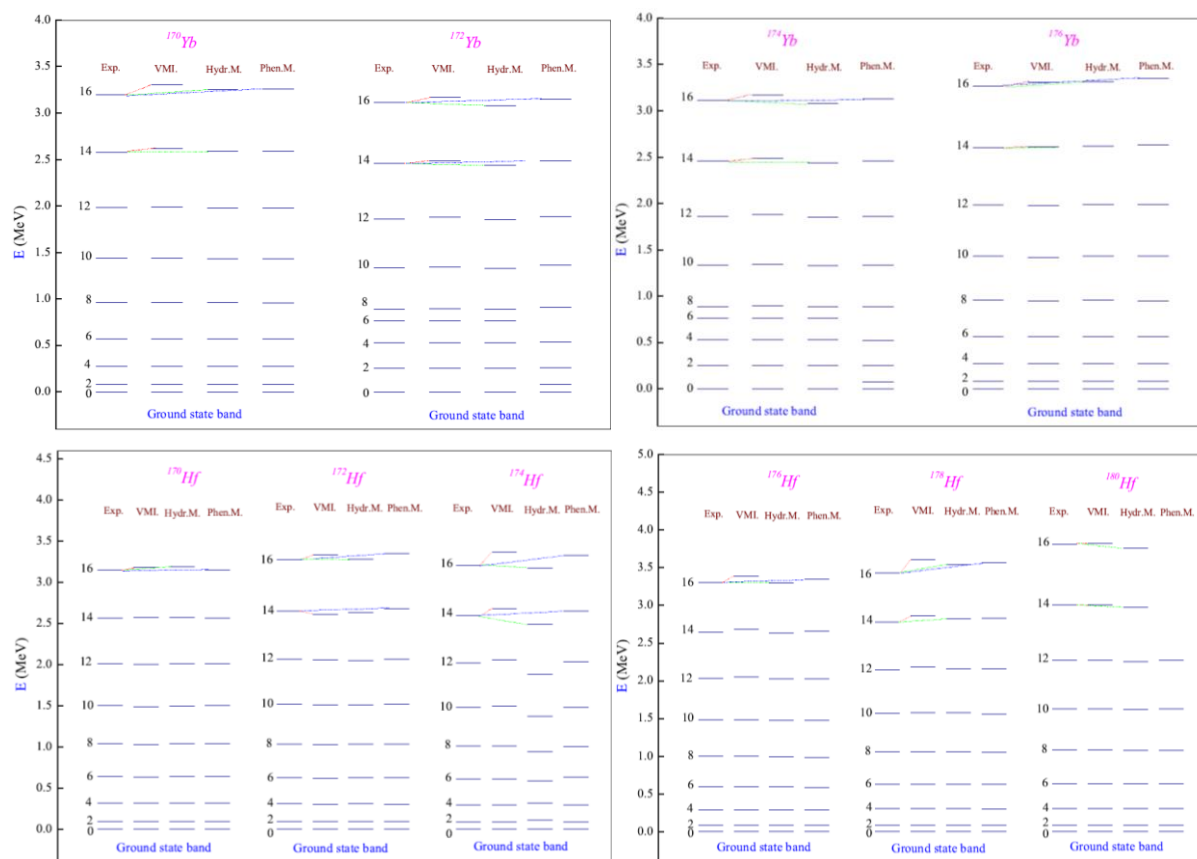


FIGURE 1. The energy spectra of rotational ground state bands for *Yb* and *Hf* isotopes

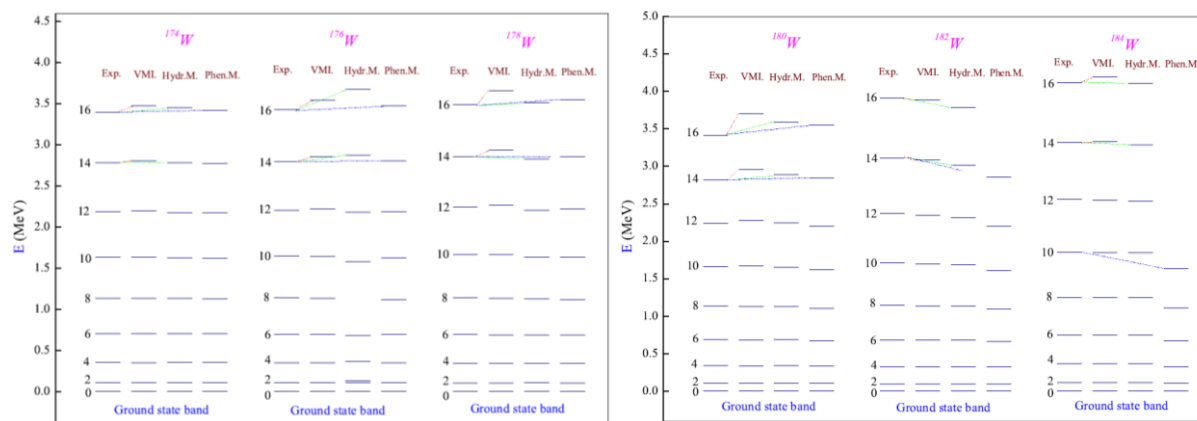


FIGURE 2. The energy spectra of rotational ground state bands of *W* isotopes

This comparison showed the results of this model are coincide with the results of VMI and phenomenological models and also good agreement with the experiment data. This coincidence means that the effect of such factors on the nuclei under consideration and for the energy levels less than 12 is small, that is the energy levels for considered nuclei are less than the energy required to split the nucleon pairs.

A graphical comparison of the calculated results and experimental data of the energy ratios of the excited states  $E_1/E_2$  as a function of  $R = E_4/E_2$  in the range

2.90 to 3.33 for all  $I$  up to 20 is displayed in Figure 3. It is shown clearly that the coincidence between experimental data and the predictions of the current work for energy  $I < 14$  has a good agreement, that is theoretical curve passes nearly through all experimental points. For  $I < 14$  there are small and systematic deviation differ from one nucleus to another and increases rapidly with the increasing  $I$ . This deviation appears as a dispersion of the experimental points around the theoretical curve. Since this deviation differ according to the nucleus, it is related to microscopic structure of the nucleus.

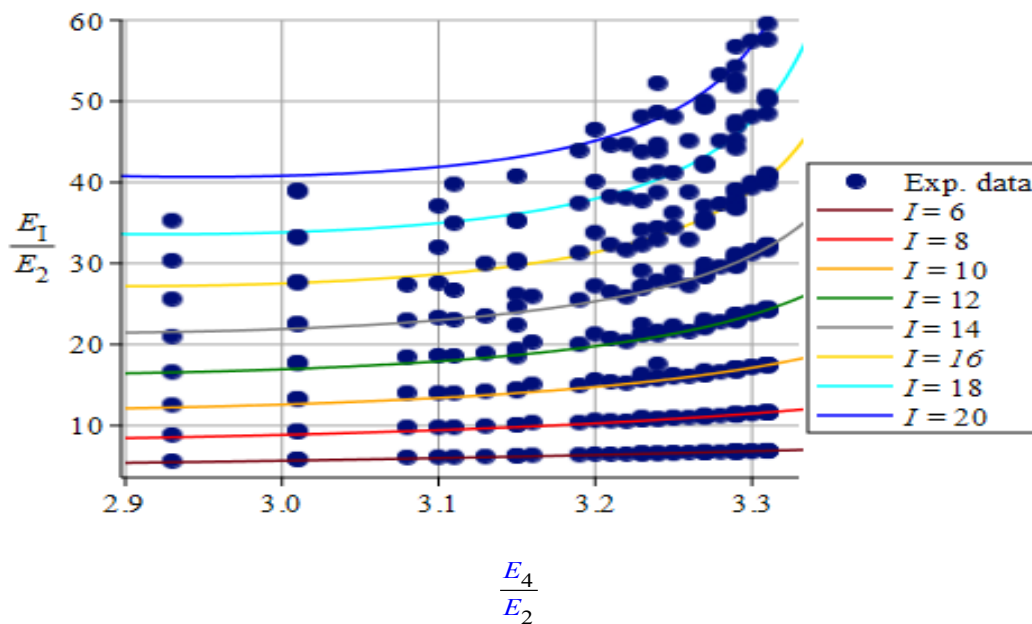


FIGURE 3. The energy ratio  $E_1/E_2$  for different values of  $I$  as a function of  $E_4/E_2$ . The solid curves are results of current work which are computed according to (21).

In Figure 4, the ratio  $E_{10}/E_2$  is plotted against  $E_4/E_2$  for the current model, the experimentally obtained data and three other models, namely the Gupta ((4) Gupta 1969), Bohr-Mottelson 2 – parameter models ((2) Bohr & Mottelson 1953), and Sood ((5) Sood 1969).

While the prediction of our work compares nicely with the experimental data points through all the region under consideration, both Gupta and Bohr-Mottelson 2 – parameter models diverge from the

experimental data except at the proximity around the  $E_4/E_2 \approx 3.33$  regime. This infers that Gupta and Bohr-Mottelson 2 – parameter model can hold only a few nuclei which is confined in the range of  $3.25 < E_4/E_2 < 3.33$ .

Although the success of Sood is impressive, it has many shortcomings nonetheless, for example a parameter  $N$  is introduced to (5) without scientific justification, nothing has been mentioned about the physical meaning or the

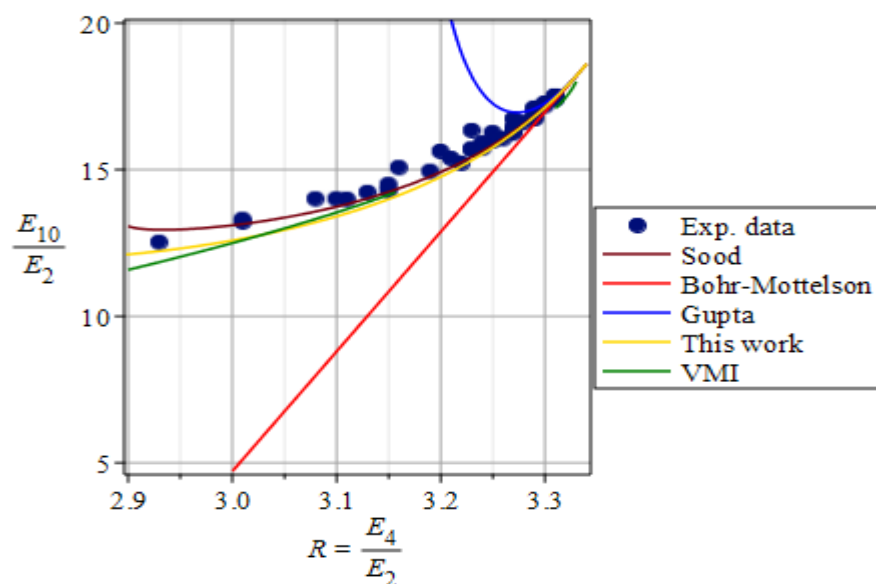


FIGURE 4. A comparison of the current work with various theoretical results and with experimental data. Red curve is from the Bohr-Mottelson 2 parameters formula, green curve is VMI model, yellow curve is current work, brown curve is Sood model (Sood 1967), blue curve is from the Gupta model (Gupta 1969) (based on 3 terms in the Gupta series), closed circles represent the experimental results and the yellow one represents this work

identity of this parameter in spite of its importance in fitting the experimental points.

Figure 5 shows the increasing behavior of the ratio  $E_{10}/J_1/J_0$  for three different nuclei as a function of  $I$ .

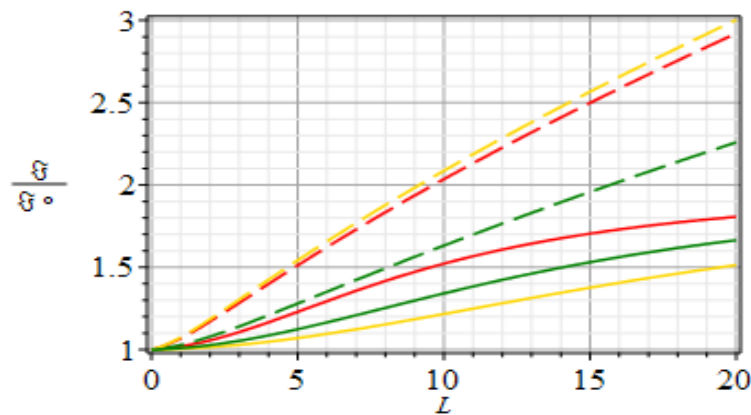


FIGURE 5. The ration  $J_1/J_0$  as a function in  $I$  for three different nuclei. The solid curves are the results of current work and dashed curves are VMI model results. The red curve is for  $^{154}\text{Gd}$ , the green curve is for  $^{160}\text{Er}$ , and the yellow curve is for  $^{190}\text{Os}$  nuclei correspondingly

The dashed curves are drawn on the basis of VMI model where we have used (8) and (11) mentioned in Mariscotti et al. (1969) together with the values of the parameter  $\sigma$  listed in Table 1 (Mariscotti et al. 1969). The solid curves in Figure 5 represent the current work. They are drawn using (20) in this work. Two features should be considered in this figure; first, the VMI model predicts that this ratio increases without limit as  $I$  increases, i.e., at high  $I$ , it can be more than 2 while, according to our results this ratio cannot exceed 2, i.e.  $J_I / J_0 = 2$  is a horizontal asymptotic to our curves. Second, VMI predicts increasing in  $J_I / J_0$  much rapidly than that predicted according to our work.

#### CONCLUSION

The quantization of the stretching that spinning nucleus exhibit led to a formula for the moment of inertia that can be applied successfully for all nuclei in the atomic mass range  $150 < A < 190$  with  $E_4 / E_2 \geq 2.9$ . While the classical treatment of such stretching works well only for nuclei in the region of strongly deformed nuclei where  $E_4 / E_2 \geq 3.25$ . Unlike VMI where the moment of inertia was considered as a general variable, our work has a characteristic that the two parameters in our formula are clear and their physical meanings are well known.

The full coincidence ( $I < 14$ ) of our formula which is derived on the basis that the moment of inertia depends only on the deformation parameter with the results presented in references (Mariscotti et al. 1969; Okhunov et al. 2015) which have taken into consideration, in addition to deformation parameter, the pairing effect and Coriolis interaction leads to a conclusion that the contributions of the latter two parameters, i.e. pairing effect and Coriolis interaction are so small and it can be neglected in the case of  $I < 14$ . For  $I < 14$  the effect of these two factors, for a specific nucleus, increasing with  $I$  and it differs from nucleus to another. For most nuclei such deviation still small up to  $I = 16$ . It is appeared obviously that the deviation from the theoretical curve increases as we go from strongly deformed nuclei where  $E_4 / E_2 \geq 3.3$  toward harmonic nuclei  $E_4 / E_2 \sim 3.24$ .

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