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Parametric Bootstrap Confidence Interval Estimation for the Percentile and Difference between the Percentiles of Delta-Lognormal Distributions with Application to Rainfall Data in Thailand

(Anggaran Selang Keyakinan Parametrik Butstrap untuk Persentil dan Perbezaan antara Peratus Taburan Delta-Lognormal dengan Aplikasi pada Data Hujan di Thailand)

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Abstract

In Thailand, flooding often occurs during the summer monsoon when many tropical storms affect the country. The motivation of this study was to plan for and mitigate the damage caused by flooding in the future. The confidence interval (CI) for the percentile of a precipitation dataset can be used to estimate the intensity of rainfall in a particular area whereas the CI for the difference between the percentiles of two datasets can be used to compare the rainfall intensities in two areas. To this end, the performances of several approaches to estimate the CI for the percentile and difference between the percentiles of delta-lognormal distributions were constructed and compared. These estimates were constructed based on the Bayesian (BS) and parametric bootstrap (PB) approaches, as well as two fiducial generalized confidence interval (FGCI) approaches. The performances of the methods were evaluated using Monte Carlo simulation, the results of which indicate that the PB approach for both CIs performed the best in all scenarios tested. Its suitability was confirmed via two illustrative examples using daily rainfall datasets for Chiang Mai and Lampang provinces in Thailand.

Keywords: Bayesian; delta-lognormal; fiducial generalized confidence interval; parametric bootstrap; rainfall

Abstrak

Di Thailand, banjir sering berlaku semasa monsun musim panas apabila banyak ribut tropika menjejaskan negara. Motivasi kajian ini adalah untuk merancang dan mengurangkan kerosakan akibat banjir pada masa hadapan. Selang keyakinan (CI) untuk persentil set data titisan boleh digunakan untuk menganggarkan keamatan curahan hujan di kawasan tertentu manakala CI untuk perbezaan antara persentil dua set data boleh digunakan untuk membandingkan keamatan curahan hujan di dua kawasan. Untuk tujuan ini, prestasi beberapa pendekatan untuk menganggarkan CI bagi persentil dan perbezaan antara persentil taburan delta-lognormal telah dibina dan dibandingkan. Anggaran ini telah dibina berdasarkan pendekatan Bayesian (BS) dan parametrik butstrap (PB) serta dua pendekatan selang keyakinan teritlak fidusial (FGCI). Prestasi kaedah telah dinilai menggunakan simulasi Monte Carlo yang hasilnya menunjukkan bahawa pendekatan PB untuk kedua-dua CI menunjukkan prestasi terbaik dalam semua senario yang diuji. Kesesuaiannya disahkan melalui dua contoh ilustrasi menggunakan set data curahan hujan harian untuk wilayah Chiang Mai dan Lampang di Thailand.

Kata kunci: Bayesian; curahan hujan; delta-lognormal; parametrik butstrap; selang keyakinan teritlak fidusial

Introduction

One of the major occupations in Thailand is farming, and the primary source of water for agricultural production is rainfall. As it is the most important factor influencing crop growth, lack of sufficient rainfall is a major limiting factor for agricultural output. On the contrary, in the middle of the rainy season in Thailand from May until October, heavy rain in northern Thailand (the largest agricultural area in the country) can cause flooding and devastation. Thus, estimating rainfall dispersion in northern Thailand is important for water management in a sustainable manner. Daily rainfall data throughout the year in northern Thailand often contain a significant number of zero observations that follow a binomial distribution. Moreover, the remaining non-zero values are all positive and follow a lognormal distribution. Thus, by combining the two distributions, the daily rainfall data can be said to follow the delta-lognormal distribution first introduced by Aitchison and Brown (1966). The delta-lognormal distribution has been used to study rainfall dispersion by many researchers (Yosboonruang, Niwitpong & Niwitpong 2020, 2019; Maneerat, Nakjai & Niwitpong 2022; Maneerat, Niwitpong & Niwitpong 2021; Thangjai, Niwitpong & Niwitpong 2022).

Estimating the parameters of a delta-lognormal distribution is an interesting problem, and both point and interval estimation have been applied to its parameters. For instance, Maneerat, Niwitpong and Niwitpong (2021) presented Bayesian (BS) estimates of the CIs for the mean and the difference between the means of delta-lognormal distributions. Yosboonruang, Niwitpong and Niwitpong (2019) studied the dispersion of rainfall in Thailand by using BS methods to estimate the CI for the coefficient of variation of a delta-lognormal distribution. Yosboonruang and Niwitpong (2020) provided statistical inference based on the ratio of coefficients of variation of delta-lognormal distributions. To study rainfall dispersion in several areas in Thailand, Yosboonruang, Niwitpong and Niwitpong (2020) proposed BS methodology to estimate the CI for the difference between the coefficients of variation of delta-lognormal distributions and later used BS credible interval estimation for the common coefficient of variation of several delta-lognormal distributions (Yosboonruang, Niwitpong & Niwitpong 2022).

Although the mean and variance of a population are arguably the most important statistics, the percentile or quantile of the population can be more appropriate than either in some situations such as lifetime distributions because it is related to reliability functions. Thus, the 2.5th and 97.5th percentiles of a distribution can be useful tools in this endeavor. For example, in medicine, the percentile can be used to explain the majority response, which is more important than the average response in some situations. Furthermore, the percentile can be used to compare a new drug with a standard drug or predecessor. In addition, in meteorology, the difference between two percentiles can be used to study the dispersion of rainfall in two different areas or during two different time intervals. Several statisticians have constructed estimates for the CI of the population percentile or population quantile. For instance, Reiss and Ruschendorf (1976) introduced the distribution-free outer CI for the quantile. Smith and Sedransk (1983) presented the lower bounds for confidence coefficients for the CI of a finite population quantile. Md et al. (1988) provided an estimate and tested hypotheses concerning the quantile function of a normal distribution. Padgett and Tomlinson (2003) estimated the lower confidence bounds for the percentiles of Weibull and Birnbaum-Saunders distributions based on parametric bootstrap (PB) methods. Chakraborti and Li (2007) estimated the CI of the percentile of a normal distribution. Hasan and Krishnamoorthy (2018) proposed CI estimates for the mean and the percentile of zero-inflated lognormal data.

CI estimation for the difference between the percentiles of two or more populations has been provided by several researchers. For example, Mandel and Betensky (2008) proposed simultaneous CIs based on the percentile bootstrap approach. Hayter (2014) presented simultaneous CIs for several specified quantiles of an unknown distribution using their probabilities from a multinomial distribution. Balakrishnan et al. (2015) constructed CI estimates for the quantiles of a two-parameter exponential distribution under progressive type-II censoring. Malekzadeh and Jafari. (2018) tested the equality of the quantiles of two-parameter exponential distributions under progressive type-II censoring. Navruz and Özdemir (2018) presented a quantile estimation method to compare two independent groups based on the percentile bootstrap approach. Jaithun, Niwitpong and Niwitpong (2018) estimated the difference in the percentiles of two delta-lognormal independent populations. Malekzadeh and Kharrati-Kopaei (2020) constructed the simultaneous CIs for the differences between the quantiles of several two-parameter exponential distributions under a progressive type-II censoring scheme. Moreover, several researchers have used the percentile or quantile to provide statistical inference when studying rainfall data (Chen et al. 2016; Lu et al. 2013; Reis & Stedinger 2005; Serinaldi 2009; Thangjai, Niwitpong & Niwitpong 2022).

In the present study, the CIs for the percentile and the difference between the percentiles of delta-lognormal distributions are estimated using BS, PB, and two fiducial generalized confidence interval (FGCI) approaches. The BS approach is based on the prior distribution, the PB is based on the sampling distribution, and the FGCI approaches are based on the fiducial generalized pivotal quantities (FGPQs) of the parameter. The estimation methods for the CIs for the percentile and difference between percentiles of delta-lognormal distributions were applied to study rainfall data from Chiang Mai and Lampang provinces, Thailand.

CI ESTIMATION FOR THE PERCENTILE

In statistical application, the true zero value and the positive value can be observed. The data containing true zero and positive values are the delta-lognormal distribution. For true zero value, the number of true zero observed value defined by $n_{(0)}$ has a binomial distribution with the probability of zero observation $\delta' = 1-\delta$. For positive value, the number of positive observed value defined by $n_{(1)}$ has a log-normal distribution with the probability of positive observation δ. The sample of size is defined by $n = n_{(0)} + n_{(1)}$. Let $X = (X_1, X_2, ..., X_n)$ be a non-negative random sample drawn from the delta-lognormal distribution with parameters mean μ, variance σ^2, and probability of obtaining the positive observation δ. Moreover, let $x = (x_1, x_2, ..., x_n)$ be the observed value of $X = (X_1, X_2, ..., X_n)$. The distribution function of delta-lognormal distribution is denoted by

$$G(x_j;\mu,\sigma^2,\delta) = \begin{cases} \delta' & ; x_j = 0 \\ \delta' + \delta F(x_j;\mu,\sigma^2) & ; x_j > 0 \end{cases}, \quad (1)$$

where $F(x_j;\mu,\sigma^2)$ is the log-normal cumulative distribution function and $j = 1,2,...,n$.

Let $Y = \ln(X)$ be independent log-transformed lognormal random variables. Assume that $Y = (Y_1, Y_2, ..., Y_n)$ is the normal distribution with mean μ and variance σ^2. Let $\bar{Y}$ and S^2 be the estimators of mean and variance, respectively. Moreover, let $\bar{Y}_{(1)}$ and $S^2_{(1)}$ be the estimators of mean and variance based on the log-transformed positive observations. Also, $\bar{y}$, $\bar{y}_{(1)}$, s^2, and $s^2_{(1)}$ are observed values of $\bar{Y}$, $\bar{Y}_{(1)}$, S^2, and $S^2_{(1)}$, respectively.

Let q_p be the p th quantile of the delta-lognormal distribution. From Equation (1), that is $G(q_p;\mu,\sigma^2,\delta) = p$. The quantile of the delta-lognormal distribution can be written as

$$q_p = \begin{cases} 0 & ; p < \delta' \\ \exp\left(\mu + \Phi^{-1}\left(\frac{p-\delta'}{1-\delta'}\right)\sigma\right) & ; p > \delta' \end{cases}, \quad (2)$$

where Φ is the standard normal distribution function.

To simplify estimation, let $\lambda_p = \mu + \Phi^{-1}\left(\frac{p-\delta'}{1-\delta'}\right)\sigma$. The estimator of the quantile is defined by

$$\hat{q}_p = \exp(\hat{\lambda}_p) = \exp\left(\bar{Y}_{(1)} + \Phi^{-1}\left(\frac{p-\delta'}{1-\delta'}\right)S_{(1)}\right). \quad (3)$$

FGCI approach for the percentile

The generalized confidence interval (GCI) approach uses the generalized pivotal quantity (GPQ) to construct the CI and to examine the hypothesis testing. Similarly, the FGCI approach uses the FGPQ to construct the CI and to examine the hypothesis testing. The FGPQ is subclass of the GPQ.

Definition Let $Y = (Y_1, Y_2, ..., Y_n)$ be the random variable with the probability density function $f(y;\mu,\sigma^2,\delta')$, where μ, σ^2, and δ' are unknown parameters. Let $y = (y_1, y_2, ..., y_n)$ be the observed value of $Y = (Y_1, Y_2, ..., Y_n)$. In other words, $y = (y_1, y_2, ..., y_n)$ is known after the data have been collected. The FGPQ is a function of Y, y, μ, σ^2, and δ', denoted by $R(Y;y,\mu,\sigma^2,\delta')$. It satisfies the following two conditions (Hannig et al. 2006):

1. The conditional distribution of $R(Y;y,\mu,\sigma^2,\delta')$ at $Y = y$ is free of the nuisance parameter.
2. The value of $R(Y;y,\mu,\sigma^2,\delta')$ at $Y = y$ is the parameter of interest.

The $100(1-\alpha)\%$ CI for parameter of interest can be constructed using the quantile of $R(Y;y,\mu,\sigma^2,\delta')$ Therefore, the $100(1-\alpha)\%$ two-sided CI is $[R(\alpha/2),$ $R(1-\alpha/2)]$, where $R(\alpha/2)$, and $R(1-\alpha/2)]$ denote the $(\alpha/2)$-th and $(1-\alpha/2)$-th quantile of $R(Y;y,\mu,\sigma^2,\delta')$, respectively.

In this paper, we proposed two FGCIs approaches based on the difference of FGPQs. First, the FGCI approach based on fiducial quantity, denoted by FGCI1 approach, was presented. Second, the FGCI approach based on optimal generalized fiducial quantity, defined by FGCI2 approach, was showed.

For FGCI1 approach, the FGPQ for λ_p is given by

$$R_{\lambda_p} = R_\mu + \frac{\sqrt{R_{\sigma^2}}}{\sqrt{n_{(1)}}}\left(\frac{Z + \Phi^{-1}(Q_n)\sqrt{n_{(1)}}}{\sqrt{U_{(1)}}}\right), \quad (4)$$

where Z denotes the standard normal distribution; and $U_{(1)}$ denotes the chi-squared distribution with $n_{(1)} - 1$ degrees of freedom (Appendix 1).

The $100(1-\alpha)\%$ two-sided CI for λ_p based on the FGCI approach using fiducial quantity is

$$[L_{\lambda.FGCI1}, U_{\lambda.FGCI1}] = [R_{\lambda_p}(\alpha/2), R_{\lambda_p}(1-\alpha/2)], \quad (5)$$

where $R_{\lambda_p}(\alpha/2)$ and $R_{\lambda_p}(1-\alpha/2)$ denote the $100(\alpha/2)$-th and $100(\alpha/2)$-th percentiles of R_{λ_p}, respectively.

Therefore, the $100(1-\alpha)\%$ two-sided CI for the quantile based on the FGCI approach using fiducial quantity is

$$CI_{FGCI1} = [L_{FGCI1}, U_{FGCI1}] = [\exp(L_{\lambda.FGCI1}), \exp(U_{\lambda.FGCI1})], \quad (6)$$

where $L_{\lambda.FGCI1}$ and $U_{\lambda.FGCI1}$ are defined in Equation (5).

For FGCI2 approach, let $R_{\lambda_{p.m}}$ be the FGPQ for λ_p. Therefore, the FGPQ for λ_p is given by

$$R_{\lambda_{p.m}} = R_\mu + \frac{\sqrt{R_{\sigma^2}}}{\sqrt{n_{(1)}}}\left(\frac{Z+\Phi^{-1}(R_\eta)\sqrt{n_{(1)}}}{\sqrt{U_{(1)}}}\right), \quad (7)$$

where Z denotes standard normal distribution, $U_{(1)}$ denotes chi-squared distribution with $n_{(1)}-1$ degrees of freedom (Appendix 2).

The $100(1-\alpha)\%$ two-sided CI for λ_p based on the FGCI approach using the optimal generalized fiducial quantity is

$$[L_{\lambda.FGCI2}, U_{\lambda.FGCI2}] = [R_{\lambda_{p.m}}(\alpha/2), R_{\lambda_{p.m}}(1-\alpha/2)], \quad (8)$$

where $R_{\lambda_{p.m}}(\alpha/2)$ and $R_{\lambda_{p.m}}(1-\alpha/2)$ denote the $100(\alpha/2)$-th and $100(\alpha/2)$-th percentiles of $R_{\lambda_{p.m}}$, respectively.

Therefore, the $100(1-\alpha)\%$ two-sided CI for the quantile based on the FGCI approach using the optimal generalized fiducial quantity is

$$CI_{FGCI2} = [L_{FGCI2}, U_{FGCI2}] = [\exp(L_{\lambda.FGCI2}), \exp(U_{\lambda.FGCI2})], \quad (9)$$

where $L_{\lambda.FGCI2}$ and $U_{\lambda.FGCI2}$ are defined in Equation (8).

BS approach for the percentile

The BS approach uses the posterior distribution to construct the Bayesian confidence interval (BSCI). The posterior distribution for σ^2 and μ were used to estimate the posterior distribution for λ_p.

Let $\Phi^{-1}(Q_\eta)$ be the quatile function of the Q_η and let $U_{(1)}$ be the chi-squared distribution with $n_{(1)}-1$ degrees of freedom. Also, let Z be the standard normal distribution. The posterior distribution of λ_p is defined by

$$\lambda_p = \mu + \frac{\sqrt{\sigma^2}}{\sqrt{n_{(1)}}}\left(\frac{Z+\Phi^{-1}(Q_\eta)\sqrt{n_{(1)}}}{\sqrt{U_{(1)}}}\right), \quad (10)$$

where σ^2 and μ are $\sigma^2 \mid y$ and $\mu \mid \sigma^2, y$, respectively (Appendix 3).

The $100(1-\alpha)\%$ two-sided CI for λ_p based on the BS approach is

$$CI_{\lambda.BS} = [L_{\lambda.BS}, U_{\lambda.BS}], \quad (11)$$

where $L_{\lambda.BS}$ and $U_{\lambda.BS}$ denote the lower and upper limits of the shortest $100(1-\alpha)\%$ highest posterior density interval of λ_p, respectively.

Therefore, the $100(1-\alpha)\%$ two-sided CI for the quantile based on the BS approach is

$$CI_{BS} = [L_{BS}, U_{BS}] = [\exp(L_{\lambda.BS}), \exp(U_{\lambda.BS})], \quad (12)$$

where $L_{\lambda.BS}$ and $U_{\lambda.BS}$ are defined in Equation (11).

PB approach for the percentile

The PB approach uses the random sampling with replacement. Let $X^* = (X_1^*, X_2^*, ..., X_n^*)$ be the sample with replacement from $X = (X_1, X_2, ..., X_n)$. Moreover, let $x^* = (x_1^*, x_2^*, ..., x_n^*)$ be the observed values of $X^* = (X_1^*, X_2^*, ..., X_n^*)$. Suppose that $\bar{X}^*$ and S^* are the sample mean and sample standard deviation, respectively. Moreover, $\bar{x}^*$ and s^* are the observed values of $\bar{X}^*$ and S^*, repectively. Let $Y^* = \ln(X^*)$ be independent log-transformed lognormal random variables. Suppose that $\bar{Y}_{(1)}^*$ and $S_{(1)}^*$ denote the estimators of mean and standard deviation based on the log-transformed positive observations. Moreover, $\bar{y}_{(1)}^*$ and $s_{(1)}^*$ are observed values of $\bar{Y}_{(1)}^*$ and $S_{(1)}^*$, respectively.

The parametric bootstrap confidence interval (PBCI) based on the fiducial quantity were constructed. Let $\Phi^{-1}(Q_\eta^*)$ be the quatile function of Q_η^*. The estimator of λ_p^* is

$$\hat{\lambda}_p^* = \bar{Y}_{(1)}^* + \Phi^{-1}\left(Q_\eta^*\right)S_{(1)}^*, \quad (13)$$

(Appendix 4).

The sampling distribution is evaluated with m bootstrap statistics. Suppose that $\bar{\hat{\lambda}}_p^*$ and $sd(\hat{\lambda}_p^*)$ denote the mean and standard deviation of $\hat{\lambda}_p^*$, respectively. The lower and upper bounds for λ_p are defined by

$$L_{\lambda.PB} = \bar{\hat{\lambda}}_p^* - z_{1-\alpha/2}\, sd(\hat{\lambda}_p^*) \quad (14)$$

and

$$U_{\lambda.PB} = \bar{\hat{\lambda}}_p^* + z_{1-\alpha/2}\, sd(\hat{\lambda}_p^*), \quad (15)$$

where $z_{1-\alpha/2}$ is the $100(1-\alpha/2)$-th percentile of the standard normal distribution.

The $100(1-\alpha)\%$ two-sided CI for λ_p based on the PB approach is

$$\mathrm{CI}_{\lambda.PB} = [L_{\lambda.PB}, U_{\lambda.PB}] \quad , \tag{16}$$

where $L_{\lambda.PB}$ and $U_{\lambda.PB}$ are defined in Equations (14) and (15), respectively.

Therefore, the $100(1-\alpha)\%$ two-sided CI for the quantile based on the PB approach is

$$\mathrm{CI}_{PB} = [L_{PB}, U_{PB}] = [\exp(L_{\lambda.PB}), \exp(U_{\lambda.PB})], \tag{17}$$

where $L_{\lambda.PB}$ nd $U_{\lambda.PB}$ are defined in Equation (16).

CI ESTIMATION FOR THE DIFFERENCE BETWEEN THE PERCENTILES

For two populations, for first population, let $X_1 = (X_{11}, X_{12}, ..., X_{1n_1})$ be non-negative random sample of size n_1 from the delta-lognormal distribution with parameters mean μ_1, variance σ_1^2, and probability of obtaining the positive observation δ_1, where $n_1 = n_{1(0)} + n_{1(1)}$, $n_{1(0)}$ is the number of true zero observed value, and $n_{1(1)}$ is the number of positive observed value. Moreover, let $\delta_1' = 1-\delta_1$ be the probability of zero observation. For second population, let $X_2 = (X_{21}, X_{22}, ..., X_{2n_2})$ be non-negative random sample of size n_2 from the delta-lognormal distribution with parameters mean μ_2, variance σ_2^2, and probability of obtaining the positive observation δ_2, where $n_2 = n_{2(0)} + n_{2(1)}$, $n_{2(0)}$ is the number of true zero observed value, and $n_{2(1)}$ is the number of positive observed value. Moreover, let $\delta_2' = 1-\delta_2$ be the probability of zero observation.

Let $Y_1 = \ln(X_1)$ and $Y_2 = \ln(X_2)$ be independent log-transformed lognormal random variables. Let $\bar{Y}_1$, $\bar{Y}_2$, S_1^2, and S_2^2 be the estimators of mean and variance for population 1 and population 2, respectively. Moreover, let $\bar{Y}_{1(1)}$, $\bar{Y}_{2(1)}$, $S_{1(1)}^2$ and $S_{2(1)}^2$ be the estimators of mean and variance based on the log-transformed positive observations for population 1 and population 2, respectively. Also, $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_{1(1)}$, $\bar{y}_{2(1)}$, s_1^2, s_2^2, $s_{1(1)}^2$ and $s_{2(1)}^2$ are observed values of $\bar{Y}_1$, $\bar{Y}_2$, $\bar{Y}_{1(1)}$, $\bar{Y}_{2(1)}$, S_1^2, S_2^2, $S_{1(1)}^2$ and $S_{2(1)}^2$, respectively.

The estimator of the difference of quantiles is defined by

$$\hat{\theta} = \hat{q}_{p_1} - \hat{q}_{p_2} = \exp(\hat{\lambda}_{p_1} - \hat{\lambda}_{p_2}), \tag{18}$$

where $\hat{\lambda}_{p_1} = \bar{Y}_{1(1)} + \Phi^{-1}\left(\frac{p_1-\delta_1'}{1-\delta_1'}\right)S_{1(1)}$ and $\hat{\lambda}_{p_2} = \bar{Y}_{2(1)} + \Phi^{-1}\left(\frac{p_2-\delta_2'}{1-\delta_2'}\right)S_{2(1)}$.

FGCI approach for the difference between the percentiles

Here, the FGCI approaches based on fiducial quantity and optimal generalized fiducial quantity were presented. For the FGCI approach based on fiducial quantity. The FGPQs for λ_{p_1} and λ_{p_2} are given by

$$R_{\lambda_{p_1}} = R_{\mu_1} + \frac{\sqrt{R_{\sigma_1^2}}}{\sqrt{n_{1(1)}}}\left(\frac{Z_1 + \Phi^{-1}(Q_{\eta_1})\sqrt{n_{1(1)}}}{\sqrt{U_{1(1)}}}\right) \tag{19}$$

and

$$R_{\lambda_{p_2}} = R_{\mu_2} + \frac{\sqrt{R_{\sigma_2^2}}}{\sqrt{n_{2(1)}}}\left(\frac{Z_2 + \Phi^{-1}(Q_{\eta_2})\sqrt{n_{2(1)}}}{\sqrt{U_{2(1)}}}\right), \tag{20}$$

(Appendix 5).

Therefore, the FGPQ for $\lambda_{p_1} - \lambda_{p_2}$ is given by

$$R_{D.\lambda_p} = R_{\lambda_{p_1}} - R_{\lambda_{p_2}} \quad , \tag{21}$$

where $R_{\lambda_{p_1}}$ and $R_{\lambda_{p_2}}$ are defined in Equations (19) and (20), respectively.

The $100(1-\alpha)\%$ two-sided CI for $\lambda_{p_1} - \lambda_{p_2}$ based on the FGCI approach using fiducial quantity is

$$[L_{D.\lambda.FGCI1}, U_{D.\lambda.FGCI1}] = [R_{D.\lambda_p}(\alpha/2), R_{D.\lambda_p}(1-\alpha/2)] \ , \tag{22}$$

where $R_{D.\lambda_p}(\alpha/2)$ and $R_{D.\lambda_p}(1-\alpha/2)$ denote the $100(\alpha/2)$-th and $100(1-\alpha/2)$-th percentiles of $R_{D.\lambda_1}$ respectively.

Therefore, the $100(1-\alpha)\%$ two-sided CI for the difference between quantiles based on the FGCI approach using fiducial quantity is

$$\begin{aligned}\mathrm{CI}_{D.FGCI1} &= [L_{D.FGCI1}, U_{D.FGCI1}] \\ &= [\exp(L_{D.\lambda.FGCI1}), \exp(U_{D.\lambda.FGCI1})] \ ,\end{aligned} \tag{23}$$

where $L_{D.\lambda.FGCI1}$ and $U_{D.\lambda.FGCI1}$ are defined in Equation (22).

Suppose that $R_{\lambda_{p.m1}}$ and $R_{\lambda_{p.m2}}$ denote the FGPQs for λ_{p_1} and λ_{p_2}, respectively. Therefore, the FGPQs for λ_{p_1} and λ_{p_2} are given by

$$R_{\lambda_{p.m1}} = R_{\mu_1} + \frac{\sqrt{R_{\sigma_1^2}}}{\sqrt{n_{1(1)}}}\left(\frac{Z_1 + \Phi^{-1}(R_{\eta_{m1}})\sqrt{n_{1(1)}}}{\sqrt{U_{1(1)}}}\right) \tag{24}$$

and

$$R_{\lambda_{p.m2}} = R_{\mu_2} + \frac{\sqrt{R_{\sigma_2^2}}}{\sqrt{n_{2(1)}}}\left(\frac{Z_2 + \Phi^{-1}(R_{\eta_{m2}})\sqrt{n_{2(1)}}}{\sqrt{U_{2(1)}}}\right), \quad (25)$$

(Appendix 6).

Therefore, the FGPQ for $\lambda_{p_1} - \lambda_{p_2}$ is given by

$$R_{D.\lambda_{p.m}} = R_{\lambda_{p.m1}} - R_{\lambda_{p.m2}}, \quad (26)$$

where $R_{\lambda_{p.m1}}$ and $R_{\lambda_{p.m2}}$ are defined in Equations (24) and (25), respectively.

The $100(1-\alpha)\%$ two-sided CI for $\lambda_{p_1} - \lambda_{p_2}$ based on the FGCI approach using the optimal generalized fiducial quantity is

$$[L_{D.\lambda.FGCI2}, U_{D.\lambda.FGCI2}] = [R_{D.\lambda_{p.m}}(\alpha/2), R_{D.\lambda_{p.m}}(1-\alpha/2)], \quad (27)$$

where $R_{D.\lambda_{p.m}}(\alpha/2)$ and $R_{D.\lambda_{p.m}}(1-\alpha/2)$ denote the $100(\alpha/2)$-th and $100(1-\alpha/2)$-th percentiles of $R_{D.\lambda_{p.m}}$, respectively.

Therefore, the $100(1-\alpha)\%$ two-sided CI for the difference between quantiles based on the FGCI approach using the optimal generalized fiducial quantity is

$$CI_{D.FGCI2} = [L_{D.FGCI2}, U_{D.FGCI2}] = [\exp(L_{D.\lambda.FGCI2}), \exp(U_{D.\lambda.FGCI2})], \quad (28)$$

where $L_{D.\lambda.FGCI2}$ and $U_{D.\lambda.FGCI2}$ are defined in Equation (27).

BS approach for the difference between the percentiles

Let $\Phi^{-1}(Q_{\eta_1})$ and $\Phi^{-1}(Q_{\eta_2})$ be the quatile functions of Q_{η_1} and Q_{η_2}. Moreover, let $U_{1(1)}$ and $U_{2(1)}$ be the chi-squared distributions with $n_{1(1)}-1$ and $n_{2(1)}-1$ degrees of freedom. Suppose that Z_1 and Z_2 denote the standard normal distributions. The posterior distributions of λ_{p_1} and λ_{p_2} are defined by

$$\lambda_{p_1} = \mu_1 + \frac{\sqrt{\sigma_1^2}}{\sqrt{n_{1(1)}}}\left(\frac{Z_1 + \Phi^{-1}(Q_{\eta_1})\sqrt{n_{1(1)}}}{\sqrt{U_{1(1)}}}\right) \quad (29)$$

and

$$\lambda_{p_2} = \mu_2 + \frac{\sqrt{\sigma_2^2}}{\sqrt{n_{2(1)}}}\left(\frac{Z_2 + \Phi^{-1}(Q_{\eta_2})\sqrt{n_{2(1)}}}{\sqrt{U_{2(1)}}}\right), \quad (30)$$

where σ_1^2 is $\sigma_1^2 \mid y_1$, σ_2^2 is $\sigma_2^2 \mid y_2$, μ_1 is $\mu_1 \mid \sigma_1^2, y_1$, and μ_2 is $\mu_2 \mid \sigma_2^2, y_2$ (Appendix 7).

Therefore, the posterior distribution for $\lambda_{p_1} - \lambda_{p_2}$ is given by

$$\lambda_p = \lambda_{p_1} - \lambda_{p_2}, \quad (31)$$

where λ_{p_1} and λ_{p_2} are defined in Equations (29) and (30), respectively.

The $100(1-\alpha)\%$ two-sided CI for $\lambda_{p_1} - \lambda_{p_2}$ based on the BS approach is

$$CI_{D.\lambda.BS} = [L_{D.\lambda.BS}, U_{D.\lambda.BS}], \quad (32)$$

where $L_{D.\lambda.BS}$ and $U_{D.\lambda.BS}$ denote the lower and upper limits of the shortest $100(1-\alpha)\%$ highest posterior density interval of λ_p, respectively.

Therefore, the $100(1-\alpha)\%$ two-sided CI for the difference between quantiles based on the BS approach is

$$CI_{D.BS} = [L_{D.BS}, U_{D.BS}] = [\exp(L_{D.\lambda.BS}), \exp(U_{D.\lambda.BS})], \quad (33)$$

where $L_{D.\lambda.BS}$ and $U_{D.\lambda.BS}$ are defined in Equation (32).

PB approach for the difference between the percentiles

Let $x_1^* = (x_{11}^*, x_{12}^*, ..., x_{1n_1}^*)$ be the sample with replacement from $X_1 = (X_{11}, X_{12}, ..., X_{1n_1})$. Also, let $X_1^* = (X_{11}^*, X_{12}^*, ..., X_{1n_1}^*)$ be the observed values of $X_1^* = (X_{11}^*, X_{12}^*, ..., X_{1n_1}^*)$. Moreover, let $X_2^* = (X_{21}^*, X_{22}^*, ..., X_{2n_2}^*)$ be the sample with replacement from $X_2^* = (X_{21}^*, X_{22}^*, ..., X_{2n_2}^*$. Also, let $X_2^* = (X_{21}^*, X_{22}^*, ..., X_{2n_2}^*$ be the observed values of $X_2^* = (X_{21}^*, X_{22}^*, ..., X_{2n_2}^*)$.

Let $Y_1^* = \ln(X_1^*)$ and $Y_2^* = \ln(X_2^*)$ be independent log-transformed lognormal random variables. Let $\bar{Y}_1^*$, $\bar{Y}_2^*$, S_1^{2*}, and S_2^{2*} be the estimators of mean and variance for population 1 and population 2, respectively. Moreover, let $\bar{Y}_{1(1)}^*$, $\bar{Y}_{2(1)}^*$, $S_{1(1)}^{2*}$ and $S_{2(1)}^{2*}$ be the estimators of mean and variance based on the log-transformed positive observations for population 1 and population 2, respectively. Also, $\bar{y}_1^*$, $\bar{y}_2^*$, $\bar{y}_{1(1)}^*$, $\bar{y}_{2(1)}^*$, s_1^{2*}, s_2^{2*}, $s_{1(1)}^{2*}$ and $s_{2(1)}^{2*}$ are observed values of $\bar{Y}_1^*$, $\bar{Y}_2^*$, $\bar{Y}_{1(1)}^*$, $\bar{Y}_{2(1)}^*$, S_1^{2*}, S_2^{2*}, $S_{1(1)}^{2*}$ and $S_{2(1)}^{2*}$, respectively.

The PBCI for the difference of quantiles based on the fiducial quantity were estimated. Let $\Phi^{-1}(Q_{\eta_1}^*)$ and $\Phi^{-1}(Q_{\eta_2}^*)$ be the quatile functions of $Q_{\eta_1}^*$ and $Q_{\eta_2}^*$. The estimators of λ_{p_1} and λ_{p_2} are

$$\hat{\lambda}_{p_1}^* = \bar{Y}_{1(1)}^* + \Phi^{-1}\left(Q_{\eta_1}^*\right)S_{1(1)}^* \quad (34)$$

and

$$\hat{\lambda}^*_{p_2} = \bar{Y}^*_{2(1)} + \Phi^{-1}\left(Q^*_{\eta_2}\right)S^*_{2(1)}\ , \qquad (35)$$

(Appendix 8).

Therefore, the estimator for $\lambda_{p_1} - \lambda_{p_2}$ is given by

$$\hat{\lambda}^*_p = \hat{\lambda}^*_{p_1} - \hat{\lambda}^*_{p_2}, \qquad (36)$$

where $\hat{\lambda}^*_{p_1}$ and $\hat{\lambda}^*_{p_2}$ are defined in Equations (34) and (35), respectively.

Let $\bar{\hat{\lambda}}^*_p$ and $sd(\hat{\lambda}^*_p)$ be the mean and standard deviation of $\hat{\lambda}^*_p$, respectively. The lower and upper bounds for $\lambda_p = \lambda_{p_1} - \lambda_{p_2}$ are defined by

$$L_{D.\lambda.PB} = \bar{\hat{\lambda}}^*_p - z_{1-\alpha/2} sd(\hat{\lambda}^*_p) \qquad (37)$$

and

$$U_{D.\lambda.PB} = \bar{\hat{\lambda}}^*_p + z_{1-\alpha/2} sd(\hat{\lambda}^*_p), \qquad (38)$$

where $z_{1-\alpha/2}$ is the $100(1-\alpha/2)$-th percentile of the standard normal distribution.

The $100(1-\alpha)\%$ two-sided CI for $\lambda_{p_1} - \lambda_{p_2}$ based on the PB approach is

$$CI_{D.\lambda.PB} = [L_{D.\lambda.PB}, U_{D.\lambda.PB}]\ , \qquad (39)$$

where $L_{D.\lambda.PB}$ and $U_{D.\lambda.PB}$ are defined in Equations (37) and (38), respectively.

Therefore, the $100(1-\alpha)\%$ two-sided CI for the difference between quantiles based on the PB approach is

$$CI_{D.PB} = [L_{D.PB}, U_{D.PB}] = [\exp(L_{D.\lambda.PB}), \exp(U_{D.\lambda.PB})], \qquad (40)$$

where $L_{D.\lambda.PB}$ and $U_{D.\lambda.PB}$ are defined in Equation (39).

RESULTS

The performances of all four approaches were examined via a Monte Carlo simulation study using the RStudio programming suite. The metrics used in the comparison were the coverage probability (CP) and average length (AL). In this study, the nominal confidence level was set as 0.95. Therefore, the one with the CP in the range [0.9440,0.9560] and with the shortest AL was selected as the most suitable. For each simulation, 3,000 runs were generated and 1,500 repetitions were carried out.

To test the confidence limit estimates for the percentile of a delta-lognormal distribution, data were generated with a sample size of 10, 30, 50, or 100; a population mean of 1.00; a population variance of 0.10, 0.30, 0.50, 0.70, or 1.00; and the probability of obtaining positive observations as 0.10, 0.30, or 0.50. The performances were presented in Table 1 and displayed in Figures 1 - 3. Table 1 displays the ranks of ALs of the 95% two-sided CIs for the percentile of delta-lognormal distribution from smallest to largest. From the simulation results, the PB approach produced satisfactory results in terms of the CP in all scenarios studied. Meanwhile, the CP results for the BS and two FGCI approaches were conservative because they were close to 1.00. Figures 1 - 3 present the CPs and the ALs of the CIs for the percentiles, corresponding to various sample sizes, probabilities of non-zero values, and variances, respectively. Based on the simulation results presented in Figure 1, it can be observed that the CPs of all approaches were close to 1.00 as the sample size increased. Furthermore, the ALs of all approaches decreased as the sample size increased. According to Figure 2, the CPs of all approaches were close to 1.00 as the probability of non-zero values increased. Additionally, the ALs of all approaches were greatest when the probability of a non-zero value was equal to 0.30, in comparison to the probabilities of non-zero values being equal to 0.10 and 0.50. Based on Figure 3, the CPs of all approaches were close to 1.00 as the variance increased. Furthermore, the ALs of all approaches increased with the variance.

For the difference between the percentiles of two delta-lognormal distributions, the sample sizes were set as (10,10), (30,30), (10,30), (50,50), (30,50), (100,100), or (50,100); the population means were fixed as (1.00,1.00); and the population variances were assigned as (0.50,0.50), (0.50,1.00), (1.00,1.00), (1.00,2.00), or (2.00,2.00). The performances were showed in Table 2 and displayed in Figures 4 - 6. Table 2 shows the ranks of ALs of the 95% two-sided CIs for the difference between the percentiles of delta-lognormal distributions from smallest to largest. From the simulation results, although the CPs of all four approaches were conservative, the PB approach performed better than the other approaches in terms of the shortest AL. Figures 4 - 6 present the CPs and the ALs of the CIs for the difference between the percentiles, corresponding to various sample sizes, probabilities of non-zero values, and variances, respectively. Based on Figure 4, it is evident that the CPs of all approaches were close to 1.00 as the sample sizes increased. Furthermore, the ALs of all approaches were higher for the sample sizes (10,10) and (10,30) in comparison to the other sample sizes. According to Figure 5, the CPs of all approaches were close to 1.00 as the probabilities of non-zero values increased. Additionally, the ALs of all approaches increased as the probabilities of non-zero values increased. From Figure 6, it is apparent that the CPs of all approaches were close to 1.00 as the variance increased. Moreover, the ALs of all approaches increased with the variances.

TABLE 1. The ranks of ALs of the 95% two-sided CIs for the percentile of delta-lognormal distribution

n	μ	δ	σ^2	Rank			
				CI_{FGCI1}	CI_{FGCI2}	CI_{BS}	CI_{PB}
10	1.00	0.10	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1
		0.30	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1
		0.50	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1
30	1.00	0.10	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1
		0.30	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1
		0.50	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1
50	1.00	0.10	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1
		0.30	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1
		0.50	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1

n	μ	δ		Rank			
				CI_{FGCI1}	CI_{FGCI2}	CI_{BS}	CI_{PB}
100	1.00	0.10	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1
		0.30	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1
		0.50	0.10	3	4	2	1
			0.30	3	4	2	1
			0.50	3	4	2	1
			0.70	3	4	2	1
			1.00	3	4	2	1

TABLE 2. The ranks of ALs of the 95% two-sided CIs for the difference between the percentiles of delta-lognormal distributions

(n_1,n_2)	(μ_1,μ_2)	(δ_1,δ_2)	(σ_1^2,σ_2^2)	Rank			
				$CI_{D.FGCI1}$	$CI_{D.FGCI2}$	$CI_{D.BS}$	$CI_{D.PB}$
(10,10)	(1.00,1.00)	(0.30,0.30)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	3	4	2	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	3	4	1	2
			(2.00,2.00)	2	3	1	4
		(0.30,0.50)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	3	4	2	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	3	4	1	2
			(2.00,2.00)	3	4	2	1
		(0.50,0.50)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	3	4	2	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	3	4	2	1
			(2.00,2.00)	3	4	2	1
(30,30)	(1.00,1.00)	(0.30,0.30)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	2	4	3	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	2	3	4	1
			(2.00,2.00)	3	4	2	1
		(0.30,0.50)	(0.50,0.50)	2	4	3	1
			(0.50,1.00)	3	2	4	1
			(1.00,1.00)	2	4	3	1
			(1.00,2.00)	3	1	4	2
			(2.00,2.00)	2	4	3	1
		(0.50,0.50)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	2	4	3	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	2	4	3	1
			(2.00,2.00)	3	4	2	1
(10,30)	(1.00,1.00)	(0.30,0.30)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	3	4	2	1
			(1.00,1.00)	3	4	2	1

(n_1,n_2)	(μ_1,μ_2)	(δ_1,δ_2)	(σ_1^2,σ_2^2)	Rank			
				$CI_{D.FGCI1}$	$CI_{D.FGCI2}$	$CI_{D.BS}$	$CI_{D.PB}$
			(1.00,2.00)	3	4	2	1
			(2.00,2.00)	3	4	2	1
		(0.30,0.50)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	3	4	2	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	3	4	2	1
			(2.00,2.00)	2	3	1	4
		(0.50,0.50)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	3	4	2	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	3	4	2	1
			(2.00,2.00)	3	4	2	1
(50,50)	(1.00,1.00)	(0.30,0.30)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	2	3	4	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	2	3	4	1
			(2.00,2.00)	3	4	2	1
		(0.30,0.50)	(0.50,0.50)	4	2	3	1
			(0.50,1.00)	3	1	4	2
			(1.00,1.00)	4	2	3	1
			(1.00,2.00)	3	1	4	2
			(2.00,2.00)	3	2	4	1
		(0.50,0.50)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	2	4	3	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	2	3	3	1
			(2.00,2.00)	3	4	2	1
(30,50)	(1.00,1.00)	(0.30,0.30)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	3	4	2	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	3	4	2	1
			(2.00,2.00)	3	4	2	1
		(0.30,0.50)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	4	2	3	1
			(1.00,1.00)	4	3	2	1
			(1.00,2.00)	4	2	3	1
			(2.00,2.00)	3	4	2	1
		(0.50,0.50)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	3	4	2	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	3	4	2	1
			(2.00,2.00)	3	4	2	1
(100,100)	(1.00,1.00)	(0.30,0.30)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	3	4	2	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	3	2	4	1
			(2.00,2.00)	3	4	2	1
		(0.30,0.50)	(0.50,0.50)	4	2	3	1
			(0.50,1.00)	3	1	4	2
			(1.00,1.00)	4	1	3	2
			(1.00,2.00)	2	1	4	3
			(2.00,2.00)	4	1	3	2
		(0.50,0.50)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	2	4	3	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	3	2	4	1
			(2.00,2.00)	3	4	2	1

(n_1,n_2)	(μ_1,μ_2)	(δ_1,δ_2)	(σ_1^2,σ_2^2)	Rank			
				$CI_{D.FGCI1}$	$CI_{D.FGCI2}$	$CI_{D.BS}$	$CI_{D.PB}$
(50,100)	(1.00,1.00)	(0.30,0.30)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	3	4	2	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	3	4	2	1
			(2.00,2.00)	3	4	2	1
		(0.30,0.50)	(0.50,0.50)	4	3	2	1
			(0.50,1.00)	4	2	3	1
			(1.00,1.00)	4	3	2	1
			(1.00,2.00)	4	1	3	2
			(2.00,2.00)	4	2	3	1
		(0.50,0.50)	(0.50,0.50)	3	4	2	1
			(0.50,1.00)	3	4	2	1
			(1.00,1.00)	3	4	2	1
			(1.00,2.00)	3	4	2	1
			(2.00,2.00)	3	4	2	1

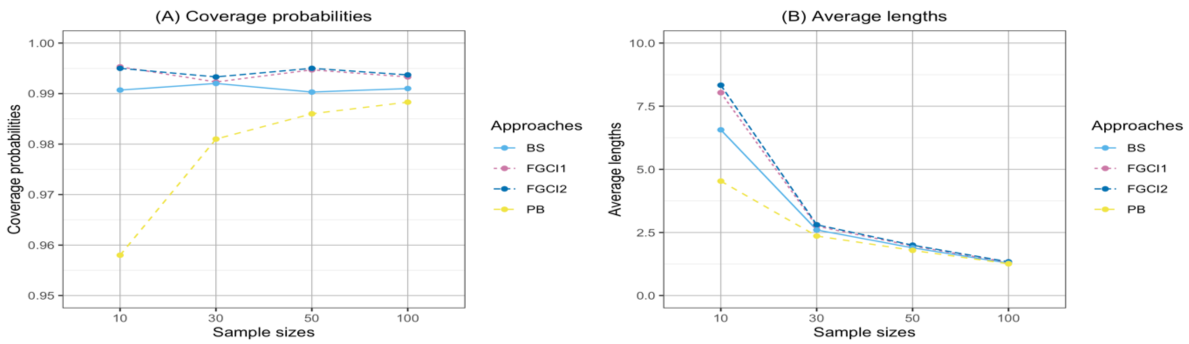

FIGURE 1. Comparison of the CPs and the ALs of the CIs for the percentile according to sample sizes

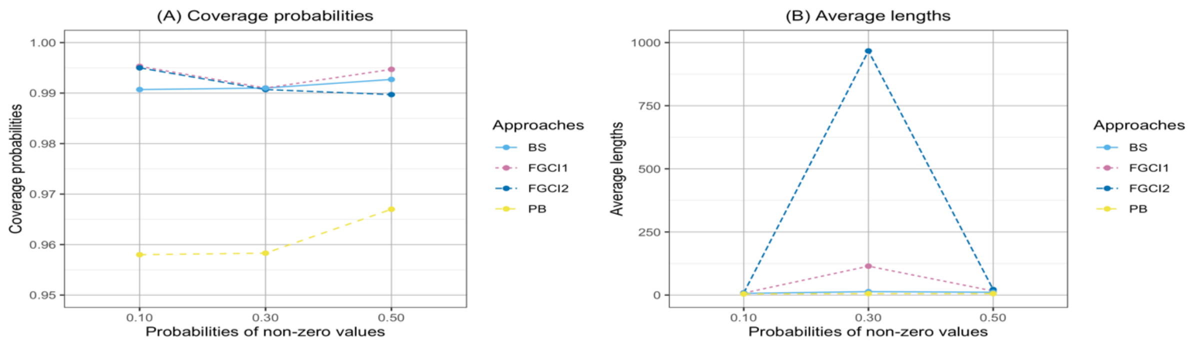

FIGURE 2. Comparison of the CPs and the ALs of the CIs for the percentile according to probabilities of non-zero values

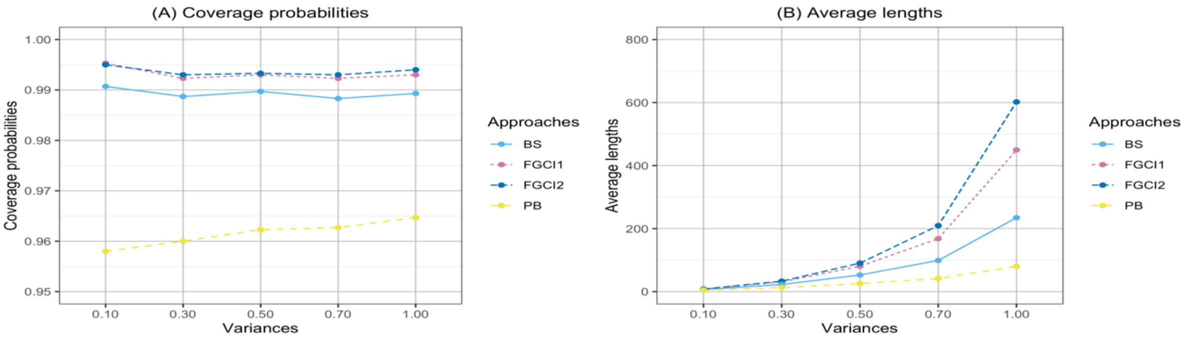

FIGURE 3. Comparison of the CPs and the ALs of the CIs for the percentile according to variances

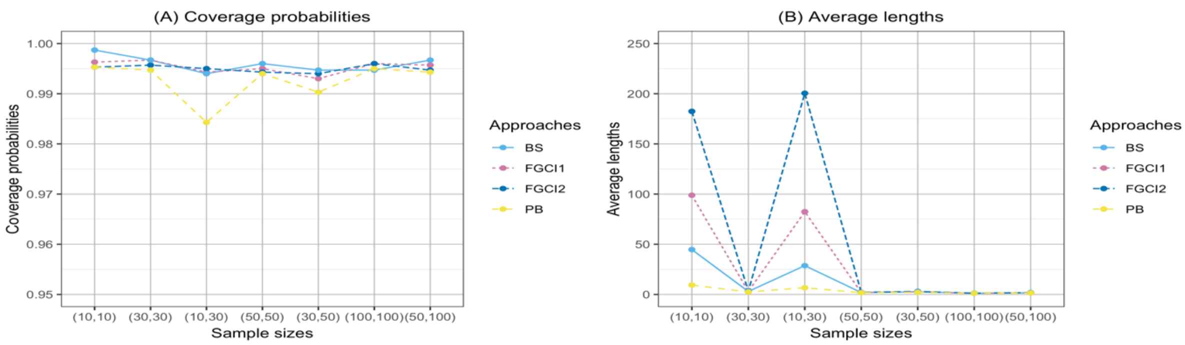

FIGURE 4. Comparison of the CPs and the ALs of the CIs for the difference between the percentiles according to sample sizes

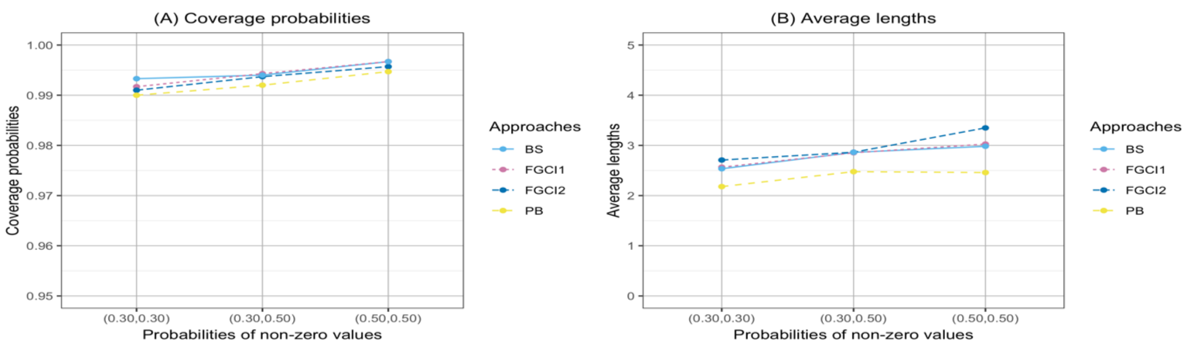

FIGURE 5. Comparison of the CPs and the ALs of the CIs for the difference between the percentiles according to probabilities of non-zero values

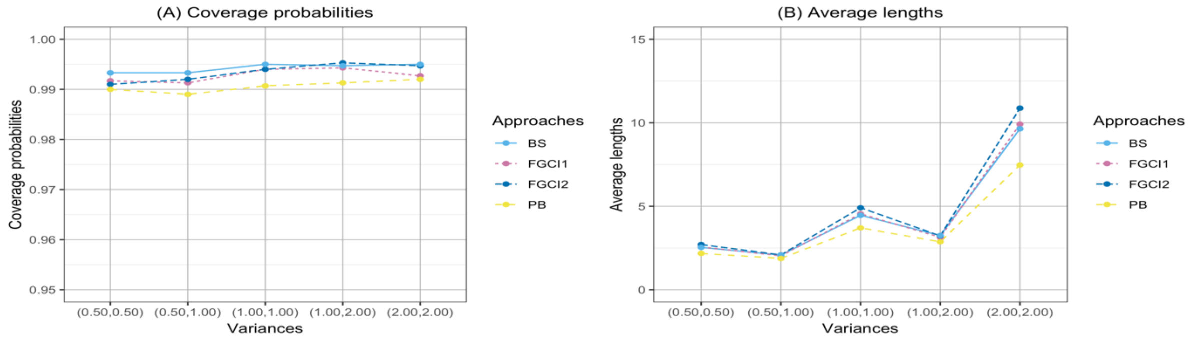

FIGURE 6. Comparison of the CPs and the ALs of the CIs for the difference between the percentiles according to variances

EMPIRICAL APPLICATION

The proposed approaches were applied to rainfall data reported by the Thai Meteorological Department in Chiang Mai and Lampang provinces collected daily from 1-25 July 2022 (Table 3). The statistics for both datasets are reported in Table 4.

Although the rainfall data consists of both zero and positive observations, the normal, lognormal, gamma, and exponential distributions can be used to model the positive observations and the Akaike information criterion (AIC) can be used to test their suitability. For the non-zero rainfall observations for both Chiang Mai and Lampang provinces, the AIC values using the lognormal distribution were the lowest (Table 5), indicating that it is the most suitable. To verify this, the histograms and the QQ-plots of the log-transformed rainfall datasets indicate that they follow normal distributions. Therefore, the delta-lognormal distribution is the most suitable for modeling the complete (zero and non-zero observations) rainfall datasets for Chiang Mai and Lampang provinces.

The 95% two-sided CI for the percentile of the rainfall data in Chiang Mai province was estimated by using the proposed approaches, which are as follows: CI_{FGCI1} = [9.2759,251.0364] with an interval length of 241.7605, CI_{FGCI2} = [10.0032,286.8422] with an interval length of 276.8390, CI_{BS} = [8.7965,216.0393] with an interval length of 207.2428, and CI_{PB} = [8.0078,132.0641] with an interval length of 124.0563. The lower and upper limits of the 95% confidence interval are the 2.50-th and 97.50-th percentiles of the rainfall data distribution in Chiang Mai province.

Similarly, the 95% two-sided CI for the percentile of the rainfall data from Lampang province was estimated by using the proposed approaches, which are as follows: CI_{FGCI1} = [7.3141,1208.6890] with an interval length of 1201.3749, CI_{FGCI2} = [10.6687,1548.6930] with an interval length of 1538.0243, CI_{BS} = [6.7923,553.5088] with an interval length of 546.7165, and CI_{PB} = [5.6766,266.0651] with an interval length of 260.3885. The lower and upper limits of the 95% confidence interval correspond to the 2.50-th and 97.50-th percentiles of the distribution of rainfall data in Lampang province.

Finally, the 95% two-sided CI estimates for the difference between the percentiles of the two rainfall datasets by using the proposed methods are $CI_{D.FGCI1}$ = [0.0370,11.4392] with an interval length of 11.4022, $CI_{D.FGCI2}$ = [0.0223,11.1240] with an interval length of 11.1017, $CI_{D.BS}$ = [0.0475,18.0370] with an interval length of 17.9895, and $CI_{D.PB}$ = [0.0756,9.1350] with an interval length of 9.0594. The lower and upper limits of the 95% confidence interval correspond to the 2.50-th and 97.50-th percentiles of the rainfall data dispersion between Chiang Mai and Lampang provinces.

It can be seen that the ALs of the PB approach were the lowest, and thus the empirical application results are in accordance with the simulation study results.

DISCUSSION

The CIs for the percentile and the difference between the percentiles of delta-lognormal distributions were constructed using two FGCI approaches, as well as BS and PB approaches. Each has its advantages when deriving the CI for the parameter of interest of a distribution (the percentile of a delta-lognormal distribution in our case). Notably, as the two FGCI approaches are based on FGPQs, their performances were similar. Moreover,

TABLE 3. The rainfall data of Chiang Mai and Lampang provinces

Province	Rainfall data (mm)				
Chiang Mai	2.0	14.2	2.6	0.3	13.3
	0.2	1.6	0.5	0.0	45.7
	0.0	10.9	18.6	0.0	7.1
	0.0	1.7	16.8	4.6	0.0
	7.7	0.5	2.0	0.3	0.8
Lampang	1.3	0.1	0.0	0.0	7.7
	0.0	0.0	1.6	0.0	23.6
	0.0	0.4	5.0	0.0	0.4
	0.0	2.8	36.6	38.3	0.0
	1.4	29.2	1.2	0.0	0.0

Source: Thai Meteorological Department (https://www.tmd.go.th/climate/climate.php)

TABLE 4. Sample statistics of Chiang Mai and Lampang provinces

Statistics	Chiang Mai province	Lampang province
n_i	25	25
$n_{i(1)}$	20	14
$n_{i(0)}$	5	11
$\bar{y}_{i(1)}$	1.02	1.11
$s^2_{i(1)}$	2.57	3.50

TABLE 5. The AIC values of Chiang Mai and Lampang provinces

Distribution	Chiang Mai province	Lampang province
Normal	154.9514	117.4884
Log-Normal	**119.5096**	**91.3952**
Gamma	121.1135	92.4712
Exponential	121.9677	95.3294

the BS approach is based on the prior distribution while the PB approach is based on the sampling distribution.

From the simulation study and empirical application results, it can be concluded that the PB approach is the best method for estimating the CIs for the percentile of a single delta-lognormal distribution and the difference between the percentiles of two delta-lognormal distributions. This conclusion is similar to those reported by Dunn (2001), Thangjai and Niwitpong (2020), Thangjai and Niwitpong (2022), and Tian et al. (2022).

CONCLUSIONS

The CIs for the percentile of a single delta-lognormal distribution and the difference between the percentiles of two delta-lognormal distributions were estimated by using two FGCI approaches, as well as the BS and PB approaches. The results of both simulation and empirical studies indicate that the PB approach performed the best in terms of the AL for both CIs. Therefore, the PB approach can be recommended to estimate the CIs for the percentile and the difference between percentiles of delta-lognormal distributions. In the future, we will provide statistical inference by using the percentiles for other distributions.

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APPENDIX 1

For FGCI1 approach, Hasan and Krishnamoorthy (2018) proposed the FGPQ based on fiducial quantity. The FGPQs for μ, σ^2, and δ' are used to estimate the FGPQ for λ_p. The FGPQ for μ as given by

$$R_\mu = \bar{y}_{(1)} - \frac{Z}{\sqrt{U_{(1)}}}\sqrt{\frac{(n_{(1)}-1)s_{(1)}^2}{n_{(1)}}} ,$$

where Z denotes standard normal distribution and $U_{(1)}$ denotes chi-squared distribution with $n_{(1)}-1$ degrees of freedom.

Moreover, the FGPQ for σ^2 is defined by

$$R_{\sigma^2} = \frac{(n_{(1)}-1)s_{(1)}^2}{U_{(1)}} ,$$

where $U_{(1)}$ denotes chi-squared distribution with $n_{(1)}-1$ degrees of freedom.

Suppose that $B_{n_{(0)}+0.5,n_{(1)}+0.5}$ denotes the beta random variable with shape parameters $n_{(0)}+0.5$ and $n_{(1)}+0.5$. Let $R_{\delta'}$ be the probability distribution of the $B_{n_{(0)}+0.5,n_{(1)}+0.5}$ random variable whose values are bounded above by p. Let V be the uniform distribution over the interval (0,1). Let p be the percentile. Let $H(p;n_{(0)}+0.5,n_{(1)}+0.5)$ be the beta distribution function with shape parameters $n_{(0)}+0.5$ and $n_{(1)}+0.5$. Suppose that

$$H^{-1}(VH(p;n_{(0)}+0.5,n_{(1)}+0.5),n_{(0)}+0.5,n_{(1)}+0.5)$$

denotes the quatile function for the beta distribution with $VH(p;n_{(0)}+0.5,n_{(1)}+0.5)$ and $n_{(0)}+0.5,n_{(1)}+0.5$. The FGPQ for δ' is given by

$$R_{\delta'} = H^{-1}(VH(p;n_{(0)}+0.5,n_{(1)}+0.5),n_{(0)}+0.5,n_{(1)}+0.5) \cdot$$

Suppose that $\Phi^{-1}(Q_\eta)$ is the quatile function of the Q_η which is defined by

$$\Phi^{-1}(Q_\eta) = \Phi^{-1}\left(\frac{p-R_{\delta'}}{1-R_{\delta'}}\right).$$

Therefore, the FGPQ for λ_p is given by

$$R_{\lambda_p} = R_\mu + \frac{\sqrt{R_{\sigma^2}}}{\sqrt{n_{(1)}}}\left(\frac{Z+\Phi^{-1}(Q_\eta)\sqrt{n_{(1)}}}{\sqrt{U_{(1)}}}\right),$$

where Z denotes the standard normal distribution and $U_{(1)}$ denotes the chi-squared distribution with $n_{(1)}-1$ degrees of freedom.

APPENDIX 2

For FGCI2 approach, Zhang et al. (2022) presented the FGPQ based on the optimal generalized fiducial quantity. Let $R_{\delta'_m}$ be the FGPQ for δ' which is given by

$$R_{\delta'_m} = \frac{H^{-1}(VH(p;n_{(0)},n_{(1)}+1),n_{(0)}+1,n_{(1)})}{2},$$

where V denotes the uniform distribution over the interval (0,1).

Suppose that $\Phi^{-1}(R_\eta)$ is the quatile function of the R_η which is defined by

$$\Phi^{-1}(R_\eta) = \Phi^{-1}\left(\frac{p-R_{\delta'_m}}{1-R_{\delta'_m}}\right).$$

Let $R_{\lambda_{p.m}}$ be the FGPQ for λ_p. Therefore, the FGPQ for λ_p is given by

$$R_{\lambda_{p.m}} = R_\mu + \frac{\sqrt{R_{\sigma^2}}}{\sqrt{n_{(1)}}}\left(\frac{Z+\Phi^{-1}(R_\eta)\sqrt{n_{(1)}}}{\sqrt{U_{(1)}}}\right),$$

where Z denotes standard normal distribution, $U_{(1)}$ denotes chi-squared distribution with $n_{(1)}-1$ degrees of freedom.

APPENDIX 3

Following Thangjai, Niwitpong and Niwitpong (2020), the posterior distribution for σ^2 is the inverse gamma distribution which is defined by

$$\sigma^2 \mid y \sim IG\left(\frac{n_{(1)}-1}{2}, \frac{(n_{(1)}-1)s^2_{(1)}}{2}\right).$$

The conditional posterior distribution for μ is the normal distribution which is defined by

$$\mu \mid \sigma^2, y \sim N(\hat{\mu}, \frac{\sigma^2}{n_{(1)}}),$$

where $\hat{\mu} = \bar{y}_{(1)}$ and σ^2 is $\sigma^2 \mid y$.

In this paper, the BSCI based on the fiducial quantity were estimated. Again, let $B_{n_{(0)}+0.5,n_{(1)}+0.5}$ be the beta random variable with shape parameters $n_{(0)}+0.5$ and $n_{(1)}+0.5$. Suppose that $Q_{\delta'} = H^{-1}(VH(p;n_{(0)}+0.5,n_{(1)}+0.5);n_{(0)}+0.5,n_{(1)}+0.5)$ is the probability distribution of the $B_{n_{(0)}+0.5,n_{(1)}+0.5}$ random variable, where V is the uniform distribution over the interval (0,1) and $H(p;n_{(0)}+0.5,n_{(1)}+0.5)$ denotes the beta distribution function with shape parameters $n_{(0)}+0.5$ and $n_{(1)}+0.5$. Define

$$Q_\eta = \frac{p-Q_{\delta'}}{1-Q_{\delta'}}.$$

Let $\Phi^{-1}(Q_\eta)$ be the quatile function of the Q_η and let $U_{(1)}$ be the chi-squared distribution with $n_{(1)}-1$ degrees of freedom. Also, let Z be the standard normal distribution. The posterior distribution of λ_p is defined by

$$\lambda_p = \mu + \frac{\sqrt{\sigma^2}}{\sqrt{n_{(1)}}}\left(\frac{Z+\Phi^{-1}(Q_\eta)\sqrt{n_{(1)}}}{\sqrt{U_{(1)}}}\right),$$

where σ^2 and μ are $\sigma^2 \mid y$ and $\mu \mid \sigma^2, y$, respectively.

APPENDIX 4

Let $B_{n_{(0)}+0.5,n_{(1)}+0.5}$ be the beta random variable with shape parameters $n_{(0)}+0.5$ and $n_{(1)}+0.5$. Suppose that $Q^*_{\delta'} = H^{-1}(WH(p;n_{(0)}+0.5,n_{(1)}+0.5);n_{(0)}+0.5,n_{(1)}+0.5)$ denotes the probability distribution of the $B_{n_{(0)}+0.5,n_{(1)}+0.5}$ random variable, where W is the uniform distribution over the interval (0,1) and $H(p;n_{(0)}+0.5,n_{(1)}+0.5)$ denotes the beta distribution function with shape parameters $n_{(0)}+0.5$ and $n_{(1)}+0.5$. Define

$$Q^*_\eta = \frac{p-Q^*_{\delta'}}{1-Q^*_{\delta'}}.$$

Let Φ^{-1} be the quatile function of Q^*_η. The estimator of λ^*_p is

$$\hat{\lambda}^*_p = \bar{Y}^*_{(1)} + \Phi^{-1}\left(Q^*_\eta\right)S^*_{(1)}.$$

APPENDIX 5

For the FGCI approach based on fiducial quantity. The FGPQs for μ_1 and μ_2 are given by

$$R_{\mu_1} = \bar{y}_{1(1)} - \frac{Z_1}{\sqrt{U_{1(1)}}}\sqrt{\frac{(n_{1(1)}-1)s^2_{1(1)}}{n_{1(1)}}}$$

and

$$R_{\mu_2} = \bar{y}_{2(1)} - \frac{Z_2}{\sqrt{U_{2(1)}}}\sqrt{\frac{(n_{2(1)}-1)s^2_{2(1)}}{n_{2(1)}}},$$

where Z_1 and Z_2 denote the standard normal distributions and $U_{1(1)}$ and $U_{2(1)}$ denote the chi-squared distributions with $n_{1(1)}-1$ and $n_{2(1)}-1$ degrees of freedom, respectively.

Moreover, the FGPQs for σ and σ_2^2 are defined by

$$R_{\sigma_1^2} = \frac{(n_{1(1)}-1)s_{1(1)}^2}{U_{1(1)}}$$

and

$$R_{\sigma_2^2} = \frac{(n_{2(1)}-1)s_{2(1)}^2}{U_{2(1)}}.$$

Suppose that $B_{n_{1(0)}+0.5,n_{1(1)}+0.5}$ denotes the beta random variable with shape parameters $n_{1(0)}+0.5$ and $n_{1(1)}+0.5$. Let $R_{\delta_1'}$ be the probability distribution of the $B_{n_{1(0)}+0.5,n_{1(1)}+0.5}$ random variable whose values are bounded above by p. Let V_1 be the uniform distribution over the interval (0,1). Let p be the percentile. Let $H(p;n_{1(0)}+0.5,n_{1(1)}+0.5)$ be the beta distribution function with shape parameters $n_{1(0)}+0.5$ and $n_{1(1)}+0.5$.

Suppose that $H^{-1}(V_1H(p;n_{1(0)}+0.5,n_{1(1)}+0.5),n_{1(0)}+ +0.5),n_{1(0)}+0.5,n_{1(1)}+0.$. denotes the quatile function for the beta distribution with $V_1H(p;n_{1(0)}+0.5,n_{1(1)}+0.5)$ and $n_{1(0)}+0.5,n_{1(1)}+0.5$. The FGPQ for δ_1' is given by

$$R_{\delta_1'} = H^{-1}(V_1H(p;n_{1(0)}+0.5,n_{1(1)}+0.5),n_{1(0)}+0.5,n_{1(1)}+0.5).$$

Similarly, let $B_{n_{2(0)}+0.5,n_{2(1)}+0.5}$ be the beta random variable with shape parameters $n_{2(0)}+0.5$ and $n_{2(1)}+0.5$. Suppose that $R_{\delta_2'}$ denotes the probability distribution of the $B_{n_{1(0)}+0.5,n_{1(1)}+0.5}$ random variable whose values are bounded above by p. Let V_2 be the uniform distribution over the interval (0,1). Let $H(p;n_{2(0)}+0.5,n_{2(1)}+0.5)$ be the beta distribution function with shape parameters $n_{2(0)}+0.5$ and $n_{2(1)}+0.5$.

Suppose that $H^{-1}(V_2H(p;n_{2(0)}+0.5,n_{2(1)}+0.5),n_{2(0)}+0.5,n_{2(1)}+0.5)$ denotes the quatile function for the beta distribution with $V_2H(p;n_{2(0)}+0.5,n_{2(1)}+0.5)$ and $n_{2(0)}+0.5,n_{2(1)}+0.5$. The FGPQ for δ_2' is given by

$$R_{\delta_2'} = H^{-1}(V_2H(p;n_{2(0)}+0.5,n_{2(1)}+0.5), n_{2(0)}+0.5,n_{2(1)}+0.5)\cdot$$

Suppose that $\Phi^{-1}(Q_{\eta_1})$ and $\Phi^{-1}(Q_{\eta_2})$ are the quatile functions which are defined by

$$\Phi^{-1}(Q_{\eta_1}) = \Phi^{-1}\left(\frac{p-R_{\delta_1'}}{1-R_{\delta_1'}}\right)$$

and

$$\Phi^{-1}(Q_{\eta_2}) = \Phi^{-1}\left(\frac{p-R_{\delta_2'}}{1-R_{\delta_2'}}\right).$$

The FGPQs for λ_{p_1} and λ_{p_2} are given by

$$R_{\lambda_{p_1}} = R_{\mu_1} + \frac{\sqrt{R_{\sigma_1^2}}}{\sqrt{n_{1(1)}}}\left(\frac{Z_1+\Phi^{-1}(Q_{\eta_1})\sqrt{n_{1(1)}}}{\sqrt{U_{1(1)}}}\right)$$

and

$$R_{\lambda_{p_2}} = R_{\mu_2} + \frac{\sqrt{R_{\sigma_2^2}}}{\sqrt{n_{2(1)}}}\left(\frac{Z_2+\Phi^{-1}(Q_{\eta_2})\sqrt{n_{2(1)}}}{\sqrt{U_{2(1)}}}\right).$$

APPENDIX 6

Let $R_{\delta_{m_1}'}$ and $R_{\delta_{m_2}'}$ be the FGPQs for δ_1' and δ_2' which are given by

$$R_{\delta_{m1}'} = \frac{H^{-1}(V_1H(p;n_{1(0)},n_{1(1)}+1),n_{1(0)}+1,n_{1(1)})}{2}$$

and

$$R_{\delta_{m2}'} = \frac{H^{-1}(V_2H(p;n_{2(0)},n_{2(1)}+1),n_{2(0)}+1,n_{2(1)})}{2},$$

where V_1 and V_2 denote the uniform distributions over the interval (0,1).

Suppose that $\Phi^{-1}(R_{\eta_{m1}})$ and $\Phi^{-1}(R_{\eta_{m2}})$ are the quatile functions of R_{η_1} and R_{η_2} which are defined by

$$\Phi^{-1}(R_{\eta_{m1}}) = \Phi^{-1}\left(\frac{p-R_{\delta_{m1}'}}{1-R_{\delta_{m1}'}}\right)$$

and

$$\Phi^{-1}(R_{\eta_{m2}}) = \Phi^{-1}\left(\frac{p-R_{\delta_{m2}'}}{1-R_{\delta_{m2}'}}\right).$$

Suppose that $R_{\lambda_{p.m1}}$ and $R_{\lambda_{p.m2}}$ denote the FGPQs for λ_{p_1} and λ_{p_2}, respectively. Therefore, the FGPQs for λ_{p_1} and λ_{p_2} are given by

$$R_{\lambda_{p.m1}} = R_{\mu_1} + \frac{\sqrt{R_{\sigma_1^2}}}{\sqrt{n_{1(1)}}}\left(\frac{Z_1+\Phi^{-1}(R_{\eta_{m1}})\sqrt{n_{1(1)}}}{\sqrt{U_{1(1)}}}\right)$$

and

$$R_{\lambda_{p.m2}} = R_{\mu_2} + \frac{\sqrt{R_{\sigma_2^2}}}{\sqrt{n_{2(1)}}}\left(\frac{Z_2+\Phi^{-1}(R_{\eta_{m2}})\sqrt{n_{2(1)}}}{\sqrt{U_{2(1)}}}\right).$$

APPENDIX 7

According to Thangjai, Niwitpong and Niwitpong (2020), the posterior distributions for σ_1^2 and σ_2^2 are the inverse gamma distributions which are defined by

$$\sigma_1^2 \mid y_1 \sim IG\left(\frac{n_{1(1)}-1}{2}, \frac{(n_{1(1)}-1)s_{1(1)}^2}{2}\right)$$

and

$$\sigma_2^2 \mid y_2 \sim IG\left(\frac{n_{2(1)}-1}{2}, \frac{(n_{2(1)}-1)s_{2(1)}^2}{2}\right).$$

The conditional posterior distributions for u_1 and u_2 are the normal distributions which are defined by

$$\mu_1 \mid \sigma_1^2, y_1 \sim N(\hat{\mu}_1, \frac{\sigma_1^2}{n_{1(1)}})$$

and

$$\mu_2 \mid \sigma_2^2, y_2 \sim N(\hat{\mu}_2, \frac{\sigma_2^2}{n_{2(1)}}),$$

where $\hat{\mu}_1 = \bar{y}_{1(1)}$, $\hat{\mu}_2 = \bar{y}_{2(1)}$, σ_1^2 is $\sigma_1^2 \mid y_1$, and σ_2^2 is $\sigma_2^2 \mid y_2$.

The BSCI for the difference of quantiles based on the fiducial quantity were constructed. Suppose that $B_{n_{1(0)}+0.5, n_{1(1)}+0.5}$ denotes the beta random variable with shape parameters $n_{1(0)}+0.5$ and $n_{1(1)}+0.5$. Let $Q_{\delta_1'} = H^{-1}(V_1 H(p; n_{1(0)}+0.5, n_{1(1)}+0.5); n_{1(0)}+0.5, n_{1(1)}+0.5)$ be the probability distribution of the $B_{n_{1(0)}+0.5, n_{1(1)}+0.5}$ random variable, where V_1 is the uniform distribution over the interval (0,1) and $H(p; n_{1(0)}+0.5, n_{1(1)}+0.5)$ is the beta distribution function with shape parameters $n_{1(0)}+0.5$ and $n_{1(1)}+0.5$. Define

$$Q_{\eta_1} = \frac{p - Q_{\delta_1'}}{1 - Q_{\delta_1'}}.$$

Similarly, let $B_{n_{2(0)}+0.5, n_{2(1)}+0.5}$ be the beta random variable with shape parameters $n_{2(0)}+0.5$ and $n_{2(1)}+0.5$ and let

$$Q_{\delta_2'} = H^{-1}(V_2 H(p; n_{2(0)}+0.5, n_{2(1)}+0.5); n_{2(0)}+0.5, n_{2(1)}+0.5)$$

be the probability distribution of the $B_{n_{2(0)}+0.5, n_{2(1)}+0.5}$ random variable, where V_2 is the uniform distribution over the interval (0,1) and $H(p; n_{2(0)}+0.5, n_{2(1)}+0.5)$ is the beta distribution function with shape parameters $n_{2(0)}+0.5$ and $n_{2(1)}+0.5$. Define

$$Q_{\eta_2} = \frac{p - Q_{\delta_2'}}{1 - Q_{\delta_2'}}.$$

Let $\Phi^{-1}(Q_{\eta_1})$ and $\Phi^{-1}(Q_{\eta_2})$ be the quatile functions of Q_{η_1} and Q_{η_2}. Moreover, let $U_{1(1)}$ and $U_{2(1)}$ be the chi-squared distributions with $n_{1(1)}-1$ and $n_{2(1)}-1$ degrees of freedom. Suppose that Z_1 and Z_2 denote the standard normal distributions. The posterior distributions of λ_{p_1} and λ_{p_2} are defined by

$$\lambda_{p_1} = \mu_1 + \frac{\sqrt{\sigma_1^2}}{\sqrt{n_{1(1)}}}\left(\frac{Z_1 + \Phi^{-1}(Q_{\eta_1})\sqrt{n_{1(1)}}}{\sqrt{U_{1(1)}}}\right)$$

and

$$\lambda_{p_2} = \mu_2 + \frac{\sqrt{\sigma_2^2}}{\sqrt{n_{2(1)}}}\left(\frac{Z_2 + \Phi^{-1}(Q_{\eta_2})\sqrt{n_{2(1)}}}{\sqrt{U_{2(1)}}}\right),$$

where σ_1^2 is $\sigma_1^2 \mid y_1$, σ_2^2 is $\sigma_2^2 \mid y_2$, μ_1 is $\mu_1 \mid \sigma_1^2, y_1$, and μ_2 is $\mu_2 \mid \sigma_2^2, y_2$.

APPENDIX 8

Let $B_{n_{1(0)}+0.5, n_{1(1)}+0.5}$ be the beta random variable with shape parameters $n_{1(0)}+0.5$ and $n_{1(1)}+0.5$.. Suppose that $Q_{\delta_1'}^* = H^{-1}(W_1 H(p; n_{1(0)}+0.5, n_{1(1)}+0.5); n_{1(0)}+0.5, n_{1(1)}+0.5)$ denotes the probability distribution of the $B_{n_{1(0)}+0.5, n_{1(1)}+0.5}$ random variable, where W_1 is the uniform distribution over the interval (0,1) and $H(p; n_{1(0)}+0.5, n_{1(1)}+0.5)$ denotes the beta distribution function with shape parameters $n_{1(0)}+0.5$ and $n_{1(1)}+0.5$. Define

$$Q_{\eta_1}^* = \frac{p - Q_{\delta_1'}^*}{1 - Q_{\delta_1'}^*}.$$

Moreover, let $B_{n_{2(0)}+0.5, n_{2(1)}+0.5}$ be the beta random variable with shape parameters $n_{2(0)}+0.5$ and $n_{2(1)}+0.5$.

Suppose that $Q_{\delta_2'}^* = H^{-1}(W_2 H(p; n_{2(0)}+0.5, n_{2(1)}+0.5); n_{2(0)}+0.5, n_{2(1)}+0.5)$ denotes the probability distribution of the $B_{n_{2(0)}+0.5, n_{2(1)}+0.5}$ random variable, where W_2 is the uniform distribution over the interval (0,1) and $H(p; n_{2(0)}+0.5, n_{2(1)}+0.5)$ denotes the beta distribution function with shape parameters $n_{2(0)}+0.5$and $n_{2(1)}+0.5$. Define

$$Q_{\eta_2}^* = \frac{p - Q_{\delta_2'}^*}{1 - Q_{\delta_2'}^*}.$$

Let $\Phi^{-1}(Q^*_{\eta_1})$ and $\Phi^{-1}(Q^*_{\eta_2})$ be the quatile functions of $Q^*_{\eta_1}$ and $Q^*_{\eta_2}$. The estimators of λ_{p_1} and λ_{p_2} are

$$\hat{\lambda}^*_{p_1} = \overline{Y}^*_{1(1)} + \Phi^{-1}\left(Q^*_{\eta_1}\right)S^*_{1(1)}$$

and

$$\hat{\lambda}^*_{p_2} = \overline{Y}^*_{2(1)} + \Phi^{-1}\left(Q^*_{\eta_2}\right)S^*_{2(1)}.$$