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Signless Laplacian Energy of Interval-Valued Fuzzy Graph and Its Applications

(Tenaga Laplacian Tanpa Tanda bagi Graf Kabur Bernilai-Selang dan Aplikasinya)

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ABSTRACT

An interval-valued fuzzy graph (IVFG) emanates from a fuzzy graph (FG) where the membership is given in interval form. This framework give the user more flexibility in dealing with fuzzy information. In this paper, the signless Laplacian matrix of an interval-valued fuzzy-directed graph is defined. The eigenvalue, spectrum, spectral radius, and energy of an interval-valued fuzzy-directed graph associated with the signless Laplacian matrix are reported. In addition, the lower bound of the signless Laplacian energy in this graph is highlighted. Finally, these tools are employed to build an algorithm that helps in solving some real live problems.

Keywords: Energy of a graph; interval-valued fuzzy graph; signless Laplacian matrix

ABSTRAK

Graf kabur bernilai-selang (GKBS) terpancar daripada graf kabur (GK) dengan keahliannya diberi dalam bentuk selang. Rangka kerja ini memberikan pengguna lebih keluwesan dalam menangani maklumat kabur. Dalam makalah ini, matriks Laplacian tanpa tanda bagi graf berarah kabur bernilai-selang ditakrifkan. Nilai eigen, spektrum, jejari spektrum dan tenaga bagi graf berarah kabur bernilai-selang yang dikaitkan dengan matriks Laplacian tanpa tanda dilaporkan. Di samping itu, sempadan bawah tenaga tanpa tanda Laplacian dalam graf ini diserlahkan. Akhir sekali, alat ini digunakan untuk membina algoritma yang membantu menyelesaikan beberapa masalah dalam kehidupan sebenar.

Kata kunci: Graf kabur bernilai-selang; matriks Laplacean tanpa tanda; tenaga graf

INTRODUCTION

Zadeh (1965) provoked the concept of a fuzzy set (FS) as an extension of a crisp set to describe the belongingness of objects to certain sensations under uncertainty. Zadeh's FS is characterized by one part called single truth membership, such that its value belongs to a closed interval [0, 1]. Zadeh again gave a new concept of an interval-valued fuzzy set (IVFS) to assign one interval truth membership for every object in a void set. This framework is more acceptable than FS because it allows the user more flexibility in organizing the data in the face of uncertainty. The IVFS has been widely studied by many researchers around the world and has been linked to many branches of mathematics, such as algebra, topology, and numerical analysis (Al-Sharqi, Ahmad & Al-Quran 2022a, 2022b, 2022c; Azam, Mamun & Nasrin 2013; Rasuli 2019). In addition, the graph theory concept was first introduced by Euler in 1736. Then, this concept developed into the determination of the energy of the graph, which was first discussed by Gutman (1978). The study of a graph has contributed to several other fields. For example, graph modeling for chemical compounds (Trinajstic 1992) includes graph energy (Gutman 1978) by determining the energy of π-electrons. This graph energy concept is also useful for predicting the boiling points, vaporization heat, and critical temperatures (Hosamani et al. 2017). Besides, the ordering index for chemical molecule coding shows a correlation with boiling points (Wang & Ma 2016). Graphs also play remarkable roles in solving network problems (Loh, Salleh & Sarmin 2014; Razak & Expert 2021).

The idea of fuzzy graphs was proposed by Rosenfeld as a generalization of Euler's graph theory. The idea of fuzzy graph theory aroused the interest of many researchers and pushed them to make a lot of contributions; for example, Mordeson and Chang-Shyh (1994) showed some operations on fuzzy graphs. The merging of these two concepts gave rise to a new idea of fuzzy graph energy by Narayanan and Mathew (2013). Wan et al. (2023) introduced and describe various methods of bipolar fuzzy graphs with their applications. Akram and Dudek (2011) investigated the cartesian product, composition, and union operations on interval fuzzy graphs with various properties. Talebi and Rashmanlou (2013) solved the decision-making application based on fuzzy graph structures. Qiang et al. (2022) proposed homomorphisms and isomorphisms of interval-valued fuzzy (IVF) graphs and focused on industry management. Patra et al. (2021) also demonstrated how an IVF graph can be used to describe an ecological system in other fields.

Furthermore, associating matrices with several types of graphs has become a very popular area of research at present. The discussion can be found in Gheisari and Ahmad (2012), Romdhini and Nawawi (2023, 2022a, 2022b), Romdhini et al. (2023, 2022). Following the trend of contributions mentioned above on fuzzy graph environment, this paper focuses on representing the signless Laplacian matrix for the IVF graph. Taking the summation of the absolute eigenvalues computed from the corresponding matrices leads us to derive the formula of the signless Laplacian energy of the IVF graph.

This paper is organized as follows. We set forth several existing definitions and introduce some basic concepts in the next section 2. Subsequently, we provide the spectra of the IVF graph accompanied by examples of computation. At the end, we summarize the findings of this study in the last section.

PRELIMINARIES

In this section, we include some basic concepts of IVF graph. Now let $\mathbb{I} = \{a{:}0 \le a \le 1\}$ be the set of all real numbers between 0 and 1 and $D\,[0,1]$ be the set of all subsets of the interval [0,1] or simply D and is defined as $D\,[0,1] = \{[a,b]: a \le b;\ a,\ b \in \mathbb{I}\}$. The addition and multiplication operations of two elements of are as follows:

Definition 2.1. (Patra et al. 2021) Let $\alpha = [\underline{\alpha},\overline{\alpha}]$ and $\beta = [\underline{\beta},\overline{\beta}]$ be two intervals in . The addition (+) and multiplication (·) between α dan β are defined below:

$$\alpha + \beta = [\underline{\alpha},\overline{\alpha}] + [\underline{\beta},\overline{\beta}] = [\max(\underline{\alpha},\underline{\beta}), \max(\overline{\alpha},\overline{\beta})],$$

$$\text{and } \alpha \cdot \beta = [\underline{\alpha},\overline{\alpha}] \cdot [\underline{\beta},\overline{\beta}] = [\min(\underline{\alpha},\underline{\beta}),\ \min(\overline{\alpha},\overline{\beta})].$$

Some properties of the IVF set are stated as follows:

1. The zero element of an IVF set is denoted by $\theta = [0,0]$.
2. The unit element of an IVF set is denoted by $\varepsilon = [1,1]$.
3. The equality $\alpha = \beta$, where $\alpha = [\underline{\alpha},\overline{\alpha}]$ and $\beta = [\underline{\beta},\overline{\beta}]$ if and only if only if $\underline{\alpha} = \underline{\beta}$ and $\overline{\alpha} = \overline{\beta}$.
4. The inequality $\alpha \le \beta$, where $\alpha = [\underline{\alpha},\overline{\alpha}]$ and $\beta = [\underline{\beta},\overline{\beta}]$ if and only if only if $\underline{\alpha} \le \underline{\beta}$ and $\overline{\alpha} \le \overline{\beta}$.
5. The inequality $\alpha < \beta$, where $\alpha = [\underline{\alpha},\overline{\alpha}]$ and $\beta = [\underline{\beta},\overline{\beta}]$ if and only if only if $\alpha \neq \beta$ and $\underline{\alpha} < \overline{\alpha}\underline{\beta} < \overline{\beta}$.

Definition 2.2. (Ju & Wang 2009) Let $G = (V, E)$ be a crisp graph with $E \subseteq V \times V$. An IVF of a graph G is a pair $\Gamma = (P, Q)$, where $P = [\mu_{\underline{P}},\ \mu_{\overline{P}}] \in$ V is an IVF set with condition $0 \le \mu_{\underline{P}}$ (x) $\le \mu_{\overline{P}}$ (x) ≤ 1 for all $x \in V$ and $Q = [\mu_{\underline{Q}},\ \mu_{\overline{Q}}] \in$ E is an IVF relation with conditions for all x, y ∈ E:

$$\mu_{\underline{Q}}(xy) \le min\{\mu_{\underline{P}}(x), \mu_{\underline{P}}(y)\} \text{ and } \mu_{\overline{Q}}(xy) \le min\{\mu_{\overline{P}}(x), \mu_{\overline{P}}(y)\}.$$

Note that P represents an IVF vertex set of V, and Q is an IVF edge set of E. We consider Γ is a simple graph without a loop and multiple edges.

Definition 2.3. (Patra et al. 2021) Let $\vec{G}$ =(V, $\vec{E}$) be a crisp directed graph with a directed edge $\vec{E} \subseteq$ V × V. An IVF of a directed graph G is a pair $\vec{\Gamma}$ =(P, $\vec{Q}$), where P=[$\mu_{\underline{P}}$, $\mu_{\overline{P}}$]∈ V is an IVF set with condition $0 \leq \mu_{\underline{P}}(x) \leq \mu_{\overline{P}}(x) \leq 1$ for all x ∈ V and $\vec{Q}$=[$\mu_{\underline{Q}}$, $\mu_{\overline{Q}}$]∈ $\vec{E}$ is an IVF relation with conditions for all x, y ∈ $\vec{E}$:

$$\mu_{\underline{Q}}(\overrightarrow{xy}) \leq min\{\mu_{\underline{P}}(x), \mu_{\underline{P}}(y)\}, \text{ and } \mu_{\overline{Q}}(\overrightarrow{xy}) \leq min\{\mu_{\overline{P}}(x), \mu_{\overline{P}}(y)\}.$$

Note that *P* represents an IVF vertex set of *V*, and Q is an IVF edge set of *E*. We consider *Γ* is a simple graph without a loop and multiple edges.

Definition 2.4. (Patra et al. 2021) Let $\vec{\Gamma}$ = (P, $\vec{Q}$) be a weighted IVF directed graph with with $|P| = n$. The weight is a function ω: $P \times P \rightarrow$ D[0,1] with $\omega_{ij} \in D[0,1]$ be the weight between vertex v_i and v_j in *P*. Then, the adjacency matrix of a directed graph $\vec{\Gamma}$ is a square matrix of order *n*, denoted by $A = [a_{ij}]$ whose (*i, j*) - entries are

$$a_{ij} = \begin{cases} \omega_{ij}, & \text{if } v_i \neq v_j \text{ and they are adjacent} \\ \theta, & \text{otherwise.} \end{cases}$$

Definition 2.5. Let $\vec{\Gamma}$ = (P, $\vec{Q}$)be a weighted IVF directed graph and *A* is an adjacency matrix. The diagonal matrix $D = [d_{ij}]$ *whose* (*i,j*) - entries are

$$d_{ij} = \begin{cases} d_{ii}, & \text{if } v_i = v_j \\ \theta, & \text{if } v_i \neq v_j, \end{cases}$$

where d_{ii} is the degree of v_i, defined as $d_{ii} = \sum\omega_{ij}$, for ω_{ij} are entries of A and $v_i v_j \in \vec{Q}$.

Now we introduce the definition of the signless Laplacian matrix for a weighted IVF directed graph as given in Definition 2.6:

Definition 2.6. Let $\vec{\Gamma}$ = (P, $\vec{Q}$) be a weighted IVF directed graph. The signless Laplacian matrix of a directed graph $\vec{\Gamma}$ is a square matrix of order *n*, denoted by $S = [s_{ij}]$ = A + D, whose entries are (*i, j*) - entries are

$$s_{ij} = \begin{cases} \omega_{ij}, & \text{if } v_i \neq v_j \text{ and they are adjacent} \\ d_{ii}, & \text{if } v_i = v_j \\ \theta, & \text{otherwise.} \end{cases}$$

Definition 2.7. Let $\vec{\Gamma}$ = (P, $\vec{Q}$)be a weighted IVF directed graph. The trace of the signless Laplacian matrix *S* of a graph $\vec{\Gamma}$, denoted by *tr*(*S*), is defined to be the sum of the main diagonal entries of *S*.

We know *S* that is a square matrix and so we can calculate the eigenvalues of *S*. This definition brings us to the energy of the graph concept. Now, we proceed to the eigenvalue definition for *S*.

Definition 2.8. Let *S* be an *n* × *n* signless Laplacian matrix of IVF directed graph $\vec{\Gamma}$, and a scalar $\lambda = [\underline{\lambda}, \overline{\lambda}]$ is an eigenvalue of *S* if there is a non-zero vector column (or row) *Y* such that $AY = \lambda Y$ (or $YA = \lambda Y$), where *Y* is known as eigenvector with respect to *λ*.

Definition 2.9. Let *S* be an *n* × *n* signless Laplacian matrix of IVF directed graph $\vec{\Gamma}$. The spectrum of $\vec{\Gamma}$ associated with *S* is defined as the list of eigenvalues of *S* and denoted by $\sigma_S(\vec{\Gamma}) = (\lambda_1, \lambda_2, ..., \lambda_n)$.

Definition 2.10. Let $\vec{\Gamma}$ = (P, $\vec{Q}$) be a weighted IVF graph $S = [s_{ij}]$ and be the signless Laplacian matrix of IVF graph $\vec{\Gamma}$, where $\lambda_1, \lambda_2, ..., \lambda_n \in D$ [0,1] are the eigenvalues of *S*. Then, the signless Laplacian energy of $\vec{\Gamma}$ is denoted by $E_S(\vec{\Gamma})$ and defined as

$$E_S(\vec{\Gamma}) = \sum_{i=1}^{n}[\underline{\lambda}_i, \overline{\lambda}_i] = [\underline{\lambda}_1 + \underline{\lambda}_2 + \cdots + \underline{\lambda}_n, \overline{\lambda}_1 + \overline{\lambda}_2 + \cdots + \overline{\lambda}_n].$$

Definition 2.11. Let $\vec{\Gamma}$ = (P, $\vec{Q}$) be a weighted IVF graph and $S = [s_{ij}]$ be an *n* × *n* signless Laplacian matrix of $\vec{\Gamma}$, where $\lambda_1, \lambda_2, ..., \lambda_n \in D$ [0,1] are the eigenvalues of *S*. Then, the signless Laplacian spectral radius of $\vec{\Gamma}$ is denoted by $\rho_S(\vec{\Gamma})$ and defined as $\rho_S(\vec{\Gamma}) = sup\{\lambda \mid \lambda \in \sigma_S(\vec{\Gamma})\} = [sup\{\underline{\lambda}_1, \underline{\lambda}_2, ..., \underline{\lambda}_n\}, sup\{\overline{\lambda}_1, \overline{\lambda}_2, ..., \overline{\lambda}_n\}]$

MAIN RESULTS

This section will present several results on the IVF energy of the IVF directed graph using corresponding the signless Laplacian matrix. We begin with two results that provide the trace information of the signless Laplacian matrix.

TTheorem 3.1. Let $A = [a_{ij}]$ be an *n* × *n* adjacency matrix of $\vec{\Gamma}$. *Let* $S = [s_{ij}]$ be the signless Laplacian matrix of $\vec{\Gamma}$, $S = D+A$ *where* $D = diag(d_{11}, d_{22}, ..., d_{nn})$. Then $tr(S) = tr(D)$.

Proof.

By Definition 2.4, the main diagonal entries of A are θ. Then for $i = j$, the entries of S are $s_{ii} = d_{ii}$, for all $1 \leq i \leq n$, and so by *Definition 2.7*, $tr(S) = tr(D)$.

Theorem 3.2. Let A = $[a_{ij}]$ be an n×n adjacency matrix of $\vec{\Gamma}$ such that there exists a non-diagonal entry that is equal to ε. If $S = [s_{ij}]$ be the signless Laplacian matrix of $\vec{\Gamma}$, $S = D + A$ where $D = diag\ (d_{11}, d_{22}, ..., d_{nn})$, then $tr(S) = \varepsilon$. Proof.
Considering there is an element $a_{pq} = \varepsilon$, for $1 \leq p,\ q \leq n$, then $d_{pp} = d_{qq} = \varepsilon$. This means D has ε element in the main diagonal entries, and by calculating the sum of d_{ii}, for all $\leq i \leq n$, we get $tr(D) = \varepsilon$. Hence, by Theorem 3.1, $tr(S) = tr(D) = \varepsilon$.

Theorems 3.1 and 3.2 can be illustrated in Example 3.1 as given below:

Example 3.1. Let $A = \begin{bmatrix} \theta & [0.3,0.8] & \theta \\ \theta & \theta & [0.4,0.7] \\ [0.2,0.5] & \varepsilon & \theta \end{bmatrix}$,

$D = \begin{bmatrix} [0.3,0.8] & \theta & \theta \\ \theta & \varepsilon & \theta \\ \theta & \theta & [0.4,0.7] \end{bmatrix}$, we

then obtain $S = \begin{bmatrix} [0.3,0.8] & [0.3,0.8] & \theta \\ \theta & \varepsilon & [0.4,0.7] \\ [0.2,0.5] & \varepsilon & [0.4,0.7] \end{bmatrix}$.

Therefore, $tr(S) = \varepsilon = tr(D)$.

From the fact that S depends on A, we discuss several cases of A. When has a zero column or a zero row, the discussion of the eigenvalue of S is provided in Theorem 3.3.

Theorem 3.3. Let $[a_{ij}]$ be an $n \times n$ adjacency matrix of $\vec{\Gamma}$ such that the entries of a column j (or row i) of A are $a_{ij} = \theta$, for all $1 \leq i \leq n$ (or $1 \leq j \leq n$), where $\theta = [0,0]$. Let S be the signless Laplacian matrix of $\vec{\Gamma}$, $S = D +\ A$ where $D = diag\ (d_{11}, d_2, ..., d_{nn})$. Then d_{jj} (or d_{ii}) is an eigenvalue of S associated with the eigenvector Y = $[\theta\ \ \theta \ldots \varepsilon \ldots \theta)]^T$ (or Y^T), where $\varepsilon = [1,1]$ be the j-th (or i-th) entry of Y. Proof.

Suppose that $Y = [y_1\ y_2 \ldots\ y_n]^T = [\theta\ \theta\ \ldots\ \varepsilon\ \ldots\ \theta]^T$, where the j-th entry is ε,

and $S = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{bmatrix}$. By Definition 2.6, we then obtain

$$S = A + D = \begin{bmatrix} \theta & a_{12} & \cdots & \theta & \cdots & a_{1n} \\ a_{21} & \theta & \cdots & \theta & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & \theta & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \theta & \cdots & \theta \end{bmatrix} +$$

$$\begin{bmatrix} d_{11} & \theta & \cdots & \theta \\ \theta & d_{22} & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta & \theta & \cdots & d_{nn} \end{bmatrix} = \begin{bmatrix} d_{11} & a_{12} & \cdots & \theta & \cdots & a_{1n} \\ a_{21} & d_{22} & \cdots & \theta & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & d_{jj} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \theta & \cdots & d_{nn} \end{bmatrix}.$$

Since $y_k = \theta$, for $k \neq j$ and $1 \leq k \leq n$, then $s_{ik} \cdot y_k = \theta$. Meanwhile, for the $k = j$ case, we have $y_j = \varepsilon$ and so $s_{ij} \cdot y_j = s_{ij}$. This implies that

$$SY = \begin{bmatrix} \sum_{k=1}^{n}(s_{1k} \cdot y_k) \\ \sum_{k=1}^{n}(s_{2k} \cdot y_k) \\ \vdots \\ \sum_{k=1}^{n}(s_{nk} \cdot y_k) \end{bmatrix} = \begin{bmatrix} \theta \\ \vdots \\ d_{jj} \\ \vdots \\ \theta \end{bmatrix} = d_{jj} \begin{bmatrix} \theta \\ \vdots \\ \varepsilon \\ \vdots \\ \theta \end{bmatrix} = d_{jj}Y. \quad (1)$$

From Equation (1) and Definition 2.8, we conclude that d_{jj} is an eigenvalue of S. In the same manner, when A consists of a zero row and the proof is similar.

Example 3.2. Let $A = \begin{bmatrix} \theta & \theta & [0.2,0.5] \\ [0.4,0.6] & \theta & \theta \\ \theta & \theta & \theta \end{bmatrix}$,

$D = \begin{bmatrix} [0.4,0.6] & \theta & \theta \\ \theta & [0.4,0.6] & \theta \\ \theta & \theta & [0.2,0.5] \end{bmatrix}$, we

then obtain $S = \begin{bmatrix} [0.4,0.6] & \theta & [0.2,0.5] \\ [0.4,0.6] & [0.4,0.6] & \theta \\ \theta & \theta & [0.2,0.5] \end{bmatrix}$.

Now suppose that $Y = \begin{bmatrix} \theta \\ \varepsilon \\ \theta \end{bmatrix}$, and so

$$SY = \begin{bmatrix} [0.4,0.6] & \theta & [0.2,0.5] \\ [0.4,0.6] & [0.4,0.6] & \theta \\ \theta & \theta & [0.2,0.5] \end{bmatrix} \begin{bmatrix} \theta \\ \varepsilon \\ \theta \end{bmatrix} = \begin{bmatrix} \theta \\ [0.4,0.6] \\ \theta \end{bmatrix}$$

$$= [0.4,0.6] \begin{bmatrix} \theta \\ \varepsilon \\ \theta \end{bmatrix}.$$

Therefore, [0.4,0.6] is an eigenvalue of S.

Theorem 3.4. Let A = $[a_{ij}]$ be an $n \times n$ adjacency matrix of $\vec{\Gamma}$ such that the entries of a column j (or row i) of A are $a_{ij} = \theta$, for all $1 \le i \le n$ (or $1 \le j \le n$), where $\theta = [0,0]$. Let S be the signless Laplacian matrix of $\vec{\Gamma}$, $S = D + A$ where $D = diag\,(d_{11}, d_{22}, ..., d_{nn})$ and $d_{jj} \ge d_{ii}$ (or $d\_ii \ge d_{jj}$). Then $\sigma_S(\vec{\Gamma}) = (d_{11}, d_{22}, ..., d_{nn})$ is the spectrum of $\vec{\Gamma}$.

Proof.

We suppose that $S = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{bmatrix}$, and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \theta \\ \vdots \\ \varepsilon \\ \vdots \\ \theta \end{bmatrix}$, where the j-th entry is ε. We also have A that has a zero j-column. By Definition 2.6, we then obtain

$$S = A + D = \begin{bmatrix} \theta & a_{12} & \cdots & \theta & \cdots & a_{1n} \\ a_{21} & \theta & \cdots & \theta & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & \theta & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \theta & \cdots & \theta \end{bmatrix} +$$

$$\begin{bmatrix} d_{11} & \theta & \cdots & \theta \\ \theta & d_{22} & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta & \theta & \cdots & d_{nn} \end{bmatrix} = \begin{bmatrix} a_{21} & d_{22} & \cdots & \theta & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & d_{jj} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \theta & \cdots & d_{nn} \end{bmatrix}.$$

We pick $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = d_{ii}X = d_{ii}\begin{bmatrix} \theta \\ \vdots \\ \varepsilon \\ \vdots \\ \theta \end{bmatrix} = \begin{bmatrix} \theta \\ \vdots \\ d_{ii} \\ \vdots \\ \theta \end{bmatrix}$ for all $1 \le i \le n$. Since $y\_k = \theta$, for $k \ne j$ and $1 \le k \le n$, then $s_{ik} \cdot y_k = \theta$. Meanwhile, for the $k = j$ case, we have $y_j = d_{ii}$ and so $s_{ij} \cdot y_j = d_{jj} \cdot d_{ii} = d_{ii}$. This implies that

$$SY = \begin{bmatrix} \sum_{k=1}^{n}(s_{1k} \cdot y_k) \\ \sum_{k=1}^{n}(s_{2k} \cdot y_k) \\ \vdots \\ \sum_{k=1}^{n}(s_{nk} \cdot y_k) \end{bmatrix} = \begin{bmatrix} \theta \\ \theta \\ \vdots \\ d_{ii} \\ \vdots \\ \theta \end{bmatrix} = \begin{bmatrix} \theta \\ \theta \\ \vdots \\ d_{ii} \cdot d_{ii} \\ \vdots \\ \theta \end{bmatrix}$$
$$= d_{ii}\begin{bmatrix} \theta \\ \theta \\ \vdots \\ d_{ii} \\ \vdots \\ \theta \end{bmatrix} = d_{ii}Y. \tag{2}$$

From Equation (2) and Definition 2.8, we conclude that d_{ii} is an eigenvalue of S and Y is an eigenvector of S with respect to d_{ii}. The list of eigenvalues gives us the spectrum of $\vec{\Gamma}$, and the proof is similar for A consists of a zero row.

Theorem 3.5. Let $A = [a_{ij}]$ be an $n \times n$ adjacency matrix of $\vec{\Gamma}$ such that the entries of a column j (or row i) of A are $a_{ij} = \theta$, for all $1 \le i \le n$ (or $1 \le j \le n$), where $\theta = [0,0]$. Let S be the signless Laplacian matrix of $\vec{\Gamma}$, $S = D + A$ where $D = diag\,(d_{11}, d_{22}, ..., d\_{nn})$ and $d_{jj} \ge d_{ii}$ (or $d_{ii} \ge d_{jj}$). Then, the signless Laplacian energy of $\vec{\Gamma}$ is $E_S(\vec{\Gamma}) = [\underline{d}_{11} + \underline{d}_{22} + ... + \underline{d}_{nn}, \overline{d}_{11} + \overline{d}_{22} + ... + \overline{d}_{nn}]$ and $E_S(\vec{\Gamma}) \ge d_{jj}$ (or $E_S(\vec{\Gamma}) \ge d_{ii}$).

Proof.

Immediately by Theorem 3.4 and Definition 2.10, we can compute the S- energy of $\vec{\Gamma}$:

$$E_S(\vec{\Gamma}) = \textstyle\sum_{i=1}^{n}[\underline{\lambda}_i, \overline{\lambda}_i] = \sum_{i=1}^{n}[\underline{d}_{ii}, d_{ii}]$$
$$= [\underline{d}_{11} + \underline{d}_{22} + \cdots + \underline{d}_{nn},\ d_{11} + \overline{d}_{22} + \cdots + \overline{d}_{nn}].$$

The fact that $d_{jj} \ge d_{ii}$ for all $1 \le i \le n$ when A has a zero column j implies $E_S(\vec{\Gamma}) \ge d_{jj}$. In other words, d_{jj} is the lower bound of $E_S(\vec{\Gamma})$. For the second case when A consists of a zero row, since $d_{ii} \ge d_{jj}$ for all $1 \le j \le n$, therefore $E_S(\vec{\Gamma}) \ge d_{ii}$.

As an illustration of Theorems 3.4 and 3.5, we follow the matrix in Example 3.2 and the following example is obtained.

Example 3.3. By Example 3.2, with $S = \begin{bmatrix} [0.4,0.6] & \theta & [0.2,0.5] \\ [0.4,0.6] & [0.4,0.6] & \theta \\ \theta & \theta & [0.2,0.5] \end{bmatrix}$, we know that $\lambda_1 = [0.4,0.6]$ is the first eigenvalue of S. Now we need to investigate the others eigenvalues. Suppose that $Y = \begin{bmatrix} \theta \\ [0.4,0.6] \\ \theta \end{bmatrix}$, and so

$$SY = \begin{bmatrix} [0.4,0.6] & \theta & [0.2,0.5] \\ [0.4,0.6] & [0.4,0.6] & \theta \\ \theta & \theta & [0.2,0.5] \end{bmatrix}\begin{bmatrix} \theta \\ [0.4,0.6] \\ \theta \end{bmatrix}$$
$$= \begin{bmatrix} \theta \\ [0.4,0.6] \\ \theta \end{bmatrix} = [0.4,0.6]\begin{bmatrix} \theta \\ [0.4,0.6] \\ \theta \end{bmatrix}.$$

Consequently, the second eigenvalue of S is $\lambda_2 = [0.4,0.6]$ and Y is an eigenvector with respect to λ_2. Also by choosing $X = \begin{bmatrix} \theta \\ [0.2,0.5] \\ \theta \end{bmatrix}$ such that

$$SX = \begin{bmatrix} [0.4,0.6] & \theta & [0.2,0.5] \\ [0.4,0.6] & [0.4,0.6] & \theta \\ \theta & \theta & [0.2,0.5] \end{bmatrix} \begin{bmatrix} \theta \\ [0.2,0.5] \\ \theta \end{bmatrix}$$

$$= \begin{bmatrix} \theta \\ [0.2,0.5] \\ \theta \end{bmatrix} = [0.2,0.5] \begin{bmatrix} \theta \\ [0.2,0.5] \\ \theta \end{bmatrix},$$

then we get the last eigenvalue $\lambda_3 = [0.2,0.5]$ and X is an eigenvector for λ_2. Therefore, by Definition 2.9, we get the spectrum of $\vec{\Gamma}$ associated with as follows:

$$\sigma_S(\vec{\Gamma}) = (\lambda_1, \lambda_2, \lambda_3) = ([0.4,0.6],[0.4,0.6],[0.2,0.5]).$$

Now by using Definition 2.10, we can calculate the -energy of as given below:

$$E_S(\vec{\Gamma}) = \sum_{i=1}^{3}[\underline{\lambda}_i, \overline{\lambda}_i] = [0.4,0.6] + [0.4,0.6]$$

$$+ [0.2,0.5] = [1.0,1.7] \geq [0.4,0.6] = d_{22}.$$

The following two theorems discuss the eigenvalue of S when A consists of a row or column with the same entries.

Theorem 3.6. Let $A = [a_{ij}]$ be an $n \times n$ adjacency matrix of $\vec{\Gamma}$ such that the entries of a column j (or row i) of A are $a_{ij} = \alpha$, for all $1 \leq i \leq n$ (or $1 \leq j \leq n$) and $i \neq j$, otherwise $a_{ij} \leq \alpha$. Let S be the signless Laplacian matrix of Γ, S $= D + A$ where $D = diag\ (d_{11}, d_{22}, ..., d_{nn})$. Then α is an eigenvalue of S associated with the eigenvector $Y = [\varepsilon\ \ \varepsilon\ \ ...\ \ \varepsilon)]^T$ (or Y^T).

Proof.

We suppose that $Y = [y_1\ \ y_2\ \ ...\ y_n)]^T = [\varepsilon\ \ \varepsilon\ \ ...\ \ \varepsilon)]^T$, and $S = [s_{ij}]$, for $i,j = 1, 2, ..., n$.

Since α is a maximum entry of A and in a column j, $a_{ij} = \alpha$, for all $1 \leq i \leq n$ and $i \neq j$, then immediately we get $d_{jj} = \alpha$. By Definition 2.6, we then obtain

$$S = A + D = \begin{bmatrix} \theta & a_{12} & \cdots & \alpha & \cdots & a_{1n} \\ a_{21} & \theta & \cdots & \alpha & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & \theta & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \alpha & \cdots & \theta \end{bmatrix}.$$

$$+ \begin{bmatrix} d_{11} & \theta & \cdots & \theta & \cdots & \theta \\ \theta & d_{22} & \cdots & \theta & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \theta & \theta & \cdots & \alpha & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \theta & \theta & \cdots & \theta & \cdots & d_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} d_{11} & a_{12} & \cdots & \alpha & \cdots & a_{1n} \\ a_{21} & d_{22} & \cdots & \alpha & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & \alpha & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \alpha & \cdots & d_{nn} \end{bmatrix}.$$

Clearly that $d_{ii} \leq \alpha$, for $1 \leq i \leq n$, conforming from the addition operation. Since $y_k = \varepsilon$, for $1 \leq k \leq n$, then based on the multiplication operation, $s_{ik} \cdot y_k = s_{ik}$, for $1 \leq i \leq n$. This implies

$$SY = \begin{bmatrix} \sum_{k=1}^{n}(s_{1k} \cdot y_k) \\ \sum_{k=1}^{n}(s_{2k} \cdot y_k) \\ \vdots \\ \sum_{k=1}^{n}(s_{nk} \cdot y_k) \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} s_{1k} \\ \sum_{k=1}^{n} s_{2k} \\ \vdots \\ \alpha + d_{jj} \\ \vdots \\ \sum_{k=1}^{n} s_{nk} \end{bmatrix}. \quad (3)$$

Note that every row of S has element α which is the maximum interval-valued entry, then $\sum_{k=1}^{n} s_{ik} = \alpha$ and $\alpha + d_{ii} = \alpha$. Therefore, we can rewrite Equation (3) as

$$SY = \begin{bmatrix} \alpha \\ \alpha \\ \vdots \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix} = \alpha Y. \quad (4)$$

From Equation (4) and again by Definition 2.8, we conclude that α is an eigenvalue of S, and similar proof for A consists of a row with the same entries.

Theorem 3.7. Let $A = [a_{ij}]$ be an $n \times n$ adjacency matrix of $\vec{\Gamma}$. Let S be the signless Laplacian matrix of $\vec{\Gamma}$, $S = D + A$ where $D = diag\ (d_{11}, d_{22}, ..., d_{nn})$ such that $d_{ii} = \alpha$ for all $1 \leq i \leq n$ and $a_{ij} \leq \alpha$, for all $1 \leq i, j \leq n$ and $i \neq j$. Then $tr(S) = \alpha$ and it is an eigenvalue of S associated with the eigenvector $Y = [\varepsilon\ \ \varepsilon\ \ ...\ \varepsilon]^T$.

Proof.

We suppose that $Y = y_1\ \ y_2\ \ ...\ \ y_n]^T = [\varepsilon\ \ \varepsilon\ \ ...\ \varepsilon)]^T$, and $S = [s_{ij}]$, for $i, j = 1, 2, ..., n$.

By Definition 2.6, we then obtain

$$S = A + D = \begin{bmatrix} \theta & a_{12} & \cdots & a_{1n} \\ a_{21} & \theta & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \theta \end{bmatrix} + \begin{bmatrix} d_{11} & \theta & \cdots & \theta \\ \theta & d_{22} & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta & \theta & \cdots & d_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} d_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & d_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & d_{nn} \end{bmatrix}.$$

Now, since $y_k = \varepsilon$, for every $1 \leq k \leq n$, then $s_{ik} \cdot y_k = s_{ik}$, for all $1 \leq i \leq n$. This implies that

$$SY = \begin{bmatrix} \sum_{k=1}^{n}(s_{1k} \cdot y_k) \\ \sum_{k=1}^{n}(s_{2k} \cdot y_k) \\ \vdots \\ \sum_{k=1}^{n}(s_{nk} \cdot y_k) \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} s_{1k} \\ \sum_{k=1}^{n} s_{2k} \\ \vdots \\ \sum_{k=1}^{n} s_{nk} \end{bmatrix}. \quad (5)$$

Since $d_{ii} = \alpha$, for $1 \leq i \leq n$, and $a_{ij} \leq \alpha$, for all $1 \leq i, j \leq n$ and $i \neq j$, then $\sum_{k=1}^{n} s_{ik} = d_{ii} = \alpha$, for all $1 \leq i \leq n$. Then Equation (5) can be written as

$$SY = \begin{bmatrix} \alpha \\ \vdots \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix} = \alpha Y. \tag{5}$$

Consequently, α is n eigenvalue of S. Moreover, it can be seen that $tr(S) = \sum_{i=1}^{n} d_{ii} = \alpha$..

Theorem 3.8. Let $A = [a_{ij}]$ be an $n \times n$ adjacency matrix of $\vec{\Gamma}$ with $a_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$ such that $max\{\underline{a}_{i1}, \underline{a}_{i2}, \ldots, \underline{a}_{in}\} = \underline{a}$ and $max\{\overline{a}_{i1}, \overline{a}_{i2}, \ldots, \overline{a}_{in}\} = \overline{a}$, for all $1 \leq i \leq n$ (or $max\{\underline{a}_{1j}, \underline{a}_{2j}, \ldots, \underline{a}_{2j}\} = \underline{a}$ and $\max\{\overline{a}_{1j}, \overline{a}_{2j}, \ldots, \overline{a}_{2j}\} = \overline{a}$, for all $1 \leq j \leq n$.). Let S be the signless Laplacian matrix of $\vec{\Gamma}$, $S = D + A$ where $D = diag\,(d_{11}, d_{22}, \ldots, d_{nn})$, then $\alpha = [\underline{a}, \overline{a}]$ is an eigenvalue of S associated with the eigenvector $Y = [\varepsilon \ \ \varepsilon \ \ \ldots \ \ \varepsilon)]^T$ (or Y^T).

Proof.

We suppose that $Y = [y_1 \ \ y_2 \ \ \ldots \ y_n\ )]^T = [\varepsilon \ \ \varepsilon \ \ \ldots \ \ \varepsilon)]^T$, and $S = [s_{ij}]$, for $i, j = 1, 2, \ldots, n$. By Definition 2.6, we then obtain

$$S = A + D = \begin{bmatrix} \theta & a_{12} & \cdots & a_{1n} \\ a_{21} & \theta & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \theta \end{bmatrix} + \begin{bmatrix} d_{11} & \theta & \cdots & \theta \\ \theta & d_{22} & \cdots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta & \theta & \cdots & d_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} d_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & d_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & d_{nn} \end{bmatrix}.$$

Now, since $y_k = \varepsilon$, for every $1 \leq k \leq n$, then $s_{ik} \cdot y_k = s_{ik}$, for all $1 \leq i \leq n$. This implies that

$$SY = \begin{bmatrix} \sum_{k=1}^{n} (s_{1k} \cdot y_k) \\ \sum_{k=1}^{n} (s_{2k} \cdot y_k) \\ \vdots \\ \sum_{k=1}^{n} (s_{nk} \cdot y_k) \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} s_{1k} \\ \sum_{k=1}^{n} s_{2k} \\ \vdots \\ \sum_{k=1}^{n} s_{nk} \end{bmatrix}. \tag{6}$$

Since $max\{\underline{a}_{i1}, \underline{a}_{i2}, \ldots, \underline{a}_{in}\} = \underline{a}$ and $max\{\overline{a}_{i1}, \overline{a}_{i2}, \ldots, \overline{a}_{in}\} = \overline{a}$ and $d_{ii} \leq \alpha$, we then obtain $max\{\underline{s}_{i1}, \underline{s}_{i2}, \ldots, \underline{s}_{in}\} = \underline{a}$ and $max\{\overline{s}_{i1}, \overline{s}_{i2}, \ldots, \overline{s}_{in}\} = \overline{a}$, for $1 \leq i \leq n$. Consequently for $1 \leq k \leq n$, $\sum_{k=1}^{n} s_{ik}, \sum_{k=1}^{n} \underline{s}_{ik}, \sum_{k=1}^{n} \overline{s}_{ik}] = [\underline{a}, \overline{a}] = \alpha$. Then Equation (6) can be written as

$$SY = \begin{bmatrix} \alpha \\ \alpha \\ \vdots \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix} = \alpha Y,$$

and α is n eigenvalue of S. In the same manner for $max\{\underline{a}_{1j}, \underline{a}_{2j}, \ldots, \underline{a}_{2j}\} = \underline{a}$ and $\overline{a}_{1j}, \overline{a}_{2j}, \ldots, \overline{a}_{2j}$,we complete the proof.

Theorem 3.9. Let S be the signless Laplacian matrix of $\vec{\Gamma}$. Then $\rho_S(\vec{\Gamma})$ is either θ *or* $= \varepsilon$.

Proof.

When θ is only the eigenvalue of S, we have $\sigma_S(\vec{\Gamma}) = \{\theta\}$, then directly $\rho_S(\vec{\Gamma}) = \theta$. Assuming now there exists $\alpha \in \sigma_S(\vec{\Gamma})$ with $\alpha \neq \theta$. It means there is an eigenvector Y with respect to α such that $SY = \alpha Y$. Suppose that we have an arbitrary β with $\alpha \leq \beta \leq \varepsilon$. From this fact, we know that $\beta \cdot \alpha = \alpha$ and $\alpha \cdot \alpha = \alpha$. Considering $S(\alpha Y) = \alpha(AY) = \alpha(\alpha Y) = (\alpha \cdot \alpha)Y = \alpha Y = (\beta \cdot \alpha)Y = \beta\,(\alpha Y)$, then $\beta \in \sigma_S(\vec{\Gamma})$. Also, we can write $S(\alpha Y) = \alpha Y = (\varepsilon \cdot \alpha)\ Y = \varepsilon(\alpha Y)$, and so $\varepsilon \in \sigma_S(\vec{\Gamma})$. Therefore, $\rho_S(\vec{\Gamma}) = \varepsilon$.

TWO APPLICATIONS IN REAL-WORLD

In this part, we present two real-world applications that can be solved with a proposed algorithm. This algorithm is based on the tools proposed in this work, so it can be used to solve problems.

Algorithm:

1. Determine the vertex $v_1, v_2, \ldots, v_n$, the edge between v_i and v_j, and the weight of every edge ω_{ij}, for $i, j = 1, 2, \ldots, n$. 2. Construct the IVF-directed graph $\vec{\Gamma} = (P, \vec{Q})$ with P $= \{v_1, v_2, \ldots, v_n\}$. 3. Analyze the degree of every vertex v_i for all v_i in P and construct matrix D of $\vec{\Gamma}$. 4. Construct the Signless Laplacian matrix S of $\vec{\Gamma}$. 5. Compute the eigenvalues of $\vec{\Gamma}$ associated to S. 6. Calculate the energy of $\vec{\Gamma}$, $E_S(\vec{\Gamma}) = [\underline{\lambda}, \overline{\lambda}]$. 7. Compute the average of $E_S(\vec{\Gamma})$ by the following formula: $Z_i = \frac{\underline{\lambda} + \overline{\lambda}}{2}$. 8. Select the largest value of Z_i.

APPLICATION IN ECOLOGICAL MODELING

Steele (1974) has presented a production food web for the North Sea and the values for yearly production ($kcal/m^2$ year) produced by the main group of organisms and then consumed by the other organism. There are 10 groups of organisms from the lowest level 'primary production' and the highest is 'yield to man'. At the second level, pelagic herbivores produce 300 $kcal/m^2$ of feces per year which is broken down into bacteria and benthic infauna which feed on fecal material. In this case, macrobenthos consumed 50 $kcal/m^2$ and meiobenthos consumed 20 $kcal/m^2$ of the total produced by pelagic herbivores.

So that the percentage of total food consumption is obtained. According to Steele (1974), the intermediate values are speculative, based on analogy and simplifying assumptions, the whole system is bounded by fairly well-determined values for the primary productivity and the fish yield. Therefore, we assume that there are tentative values of 5%. Consequently, meiobenthos and macrobenthos consume 2%-12% and 12%-22% of fecal material, respectively. With the same analysis of other organisms' levels from the secondary data (Steele, 1974), we provide the IVF directed graph as in Figure 1.

Let the set of vertex $V = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{10}\}$ with $\boldsymbol{v}_1$: yield to man; $\boldsymbol{v}_2$: large fish; $\boldsymbol{v}_3$: inverted carnivore; $\boldsymbol{v}_4$: pelagic fish; $\boldsymbol{v}_5$: demersal fish; $\boldsymbol{v}_6$: other carnivores; $\boldsymbol{v}_7$: macrobenthos; $\boldsymbol{v}_8$: meiobenthos; $\boldsymbol{v}_9$: pelagic herbivores; and $\boldsymbol{v}_{10}$: primary production. The edges of the graph are the consumption relation between two groups of organisms. The weight between the vertex $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ is the percentage of total food consumption of $\boldsymbol{v}_j$ from the production from $\boldsymbol{v}_i$. The signless Laplacian matrix of $\vec{\Gamma}$ is obtained as follows:

$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} \end{array} \\
\begin{bmatrix}
[0.72,0.82] & \theta & \theta & \theta & \theta & \theta & \theta & \theta & \theta & \theta \\
[0.18,0.28] & [0.18,0.28] & \theta & \theta & \theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & [0.68,0.78] & [0.68,0.78] & \theta & \theta & \theta & \theta & \theta & \theta \\
[0.45,0.55] & \theta & \theta & [0.68,0.78] & [0.60,0.70] & \theta & \theta & \theta & \theta & \theta \\
[0.72,0.82] & [0.18,0.28] & \theta & \theta & [1.00,1.00] & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & [1.00,1.00] & [1.00,1.00] & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & [0.04,0.14] & [0.05,0.15] & [1.00,1.00] & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta & [1.00,1.00] & [1.00,1.00] & \theta & \theta \\
\theta & \theta & [0.08,0.18] & [0.04,0.14] & \theta & \theta & [0.12,0.22] & [0.02,0.12] & [0.14,0.24] & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta & \theta & \theta & [0.14,0.24] & [0.14,0.24]
\end{bmatrix}
\end{array}
$$

In this case, 10^{th} the column entries of A are θ. Therefore, by Theorem 3.4 we have the S-spectrum of $\vec{\Gamma}$ as $\sigma_S(\vec{\Gamma}) =$

$$\left(\begin{array}{c}[0.72,0.82],[0.18,0.28],[0.18,0.28],[0.68,0.78],[1.00,1.00],[1.00,1.00],\\ [1.00,1.00],[1.00,1.00],[0.14,0.24],[0.14,0.24]\end{array}\right)$$

By Theorem 3.9, we then get the S-spectral radius of as $\vec{\Gamma}$ as $\rho_S(\vec{\Gamma}) = [\mathbf{1.00, 1.00}] = \varepsilon$. Clearly, $\rho_S(\vec{\Gamma})$ is an interval-valued that is the upper bound of all the eigenvalues of $\vec{\Gamma}$. As seen in the system in Figure 2, macrobenthos

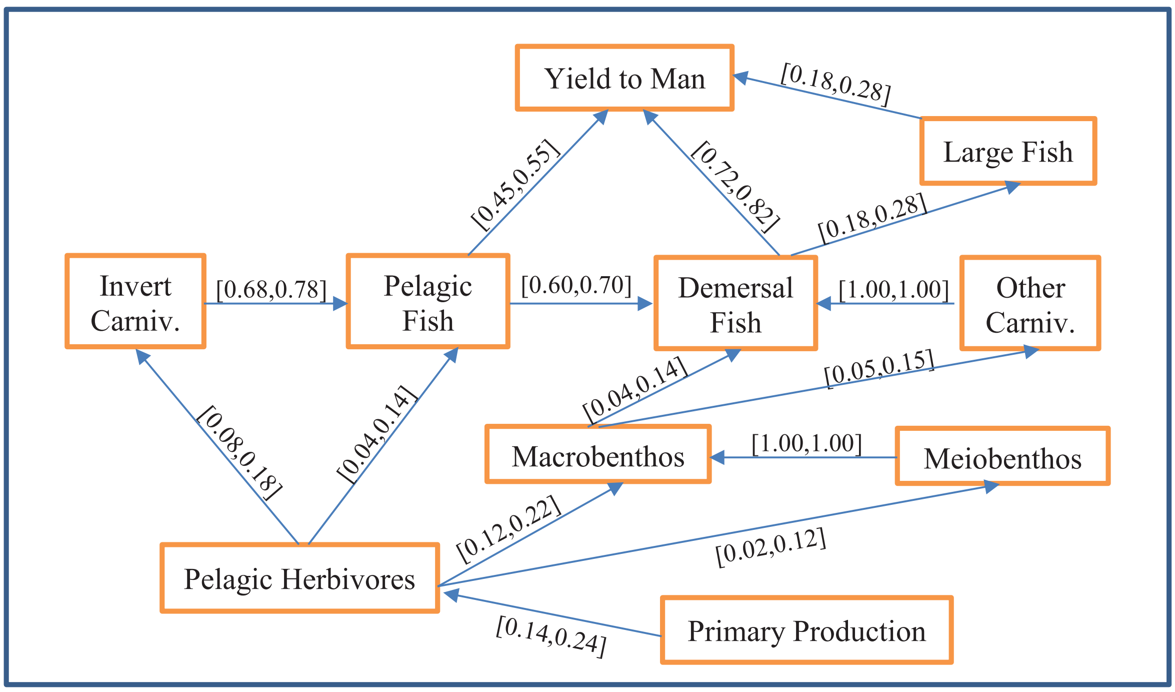

FIGURE 1. IVF directed graph $\vec{\Gamma}$ of a north sea food based on the main groups of organism

absorbs all nutrients from the meiobenthos. Moreover, the signless laplacian energy of is $\vec{\boldsymbol{\Gamma}}$ is $\boldsymbol{E_S}\,(\vec{\boldsymbol{\Gamma}}) = [\mathbf{6.04}, \mathbf{6.64}]$.

APPLICATION IN SELECTING A SUITABLE SECONDARY SCHOOL

In our daily lives, selecting the appropriate school for a son or daughter is an essential responsibility of the family, so the family takes into account many criteria for the suitable selection, including the efficiency of the teaching staff, tuition fees, the size of the school, the technological techniques used in teaching, the services provided to students in the school, such as the provision of a bus to transport students, and the cleanliness of the school.

Example 4.1. Mr. Xu wants to choose a secondary school for his daughter, and three secondary schools are available in the area where he lives. The criteria Mr. Xu is focusing on are fees, the efficiency of the teaching staff, and the technological techniques used, respectively. Mr. Xu sought the help of four of his friends (the experts) to help him choose, and accordingly, each of Mr. Xu's friends gave his opinion based on the criteria on which Mr. Xu relied, as each of them showed the difference between each criterion from one school to another and vice versa. We, in turn, analyzed the experts' opinions and organized them in the form of an IVF-directed graph ($\vec{\Gamma}$) where each IVF-directed graph represents one of these schools, as seen in Figures 3, 4, and 5, as follows.

In IVFG $\vec{\Gamma_1}$, $\vec{\Gamma_2}$ and $\vec{\Gamma_3}$ shown in Figures 3, 4, 5, the IVF number of edges represents the degree of variation of the above criteria in this school based on the opinions of each expert. As for the final degree of expert opinion, it is represented by the degree of the vertices (that we obtained after we applied Definition 2.5 to the IVF number of edges), where each vertex represents one of the four experts. For example, in IVFG $\vec{\Gamma_1}$, in Figure 3 (School 1) the IVF numbers [0.5,0,6] of expert x_1, indicates that this expert refers to selecting this school represented in Figure 3, with a degree ranging between 0.5 and 0.6. Now, to solve this problem, and to help Mr. Xu choose the appropriate school from among the three schools, we apply the above-mentioned algorithm. The signless Laplacian matrix of $\vec{\Gamma_1}$ is an 4 × 4 matrix as given below:

$$S_1 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} [0.5, 0.6] & [0.5,0.5] & [0.1,0.5] & [0.1,0.3] \\ [0.1,0.6] & [0.6,0.9] & [0.4,0.7] & [0.3,0.7] \\ [0.3,0.5] & [0.6,0.9] & [0.6,0.9] & [0.4,0.7] \\ [0.2,0.6] & [0.2,0.5] & [0.6,0.8] & [0.6,0.8] \end{bmatrix} \end{array}.$$

The S-spectrum of $\vec{\Gamma_1}$ is $\sigma_S(\vec{\Gamma_1})$, the S-spectral radius of $\vec{\Gamma_1}$ is [1.0,1.0]. Afterwards, the S-energy of $\vec{\Gamma_1}$ is $E_s(\vec{\Gamma_1})$ = [2.8,3.6].

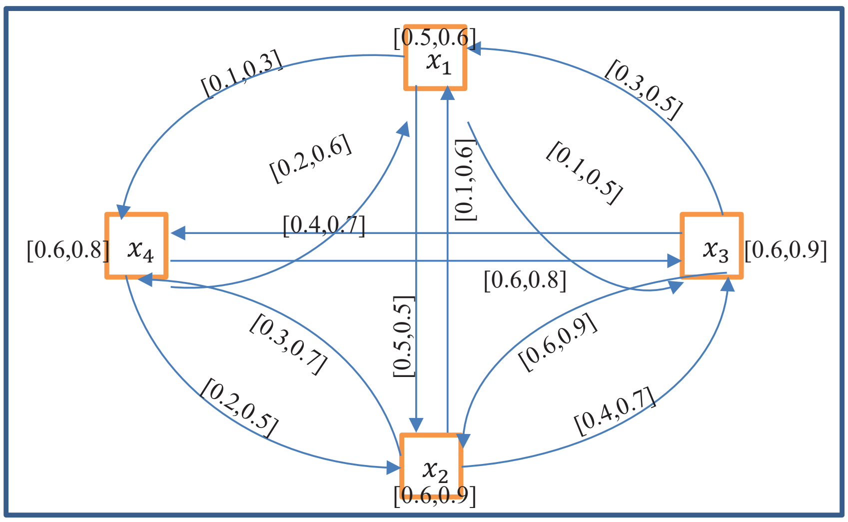

FIGURE 2. [IVF Directed Graph $\vec{\Gamma_1}$ (School 1)]

The signless Laplacian matrix of $\overrightarrow{\Gamma_2}$ is an 4×4 matrix as given herewith:

$$S_2 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} [0.5, 0.9] & [0.5,0.6] & [0.1,0.9] & [0.1,0.7] \\ [0.2,0.5] & [0.5,0.8] & [0.3,0.8] & [0.5,0.7] \\ [0.4,0.5] & [0.5,0.7] & [0.5,0.9] & [0.4,0.7] \\ [0.3,0.6] & [0.4,0.8] & [0.3,0.6] & [0.5,0.8] \end{bmatrix} \end{array}.$$

The S-spectrum of $\overrightarrow{\Gamma_2}$ is $\sigma_S(\overrightarrow{\Gamma_2})$ = ([1.0,1.0], [0.5,0.9], [0.5,0.9], [0.5,0.8]), the S-spectral radius of $\overrightarrow{\Gamma_2}$ is [1.0,1.0]. Afterwards, the S-energy of $\overrightarrow{\Gamma_2}$ is $E_S(\overrightarrow{\Gamma_2})$ = [2.5,3.6].

The signless Laplacian matrix of $\overrightarrow{\Gamma_3}$ is an 4×4 matrix as given herewith:

$$S_3 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} [0.4, 0.8] & [0.4,0.6] & [0.2,0.7] & [0.3,0.4] \\ [0.3,0.4] & [0.5,0.8] & [0.4,0.7] & [0.3,0.6] \\ [0.3,0.8] & [0.5,0.8] & [0.5,0.8] & [0.2,0.6] \\ [0.3,0.5] & [0.3,0.8] & [0.3,0.6] & [0.3,0.8] \end{bmatrix} \end{array}.$$

The S-spectrum of $\overrightarrow{\Gamma_3}$ is $\sigma_S(\overrightarrow{\Gamma_3})$ = ([1.0,1.0],[0.4,0.8],[0.5,0.8],[0.5,0.8]), the S-spectral radius of $\overrightarrow{\Gamma_3}$ is [1.0,1.0]. Afterward, the S-energy of $\overrightarrow{\Gamma_3}$ is $E_S(\overrightarrow{\Gamma_3})$ = [2.4,3.4]. Now after we apply step 7 in the above algorithm, we get $Z_1 = 3.2$, $Z_2 = 3.05$ and $Z_3 = 2.9$. Thus, according to the opinion of the four experts, the first school represented in Figure 3 is the appropriate school.

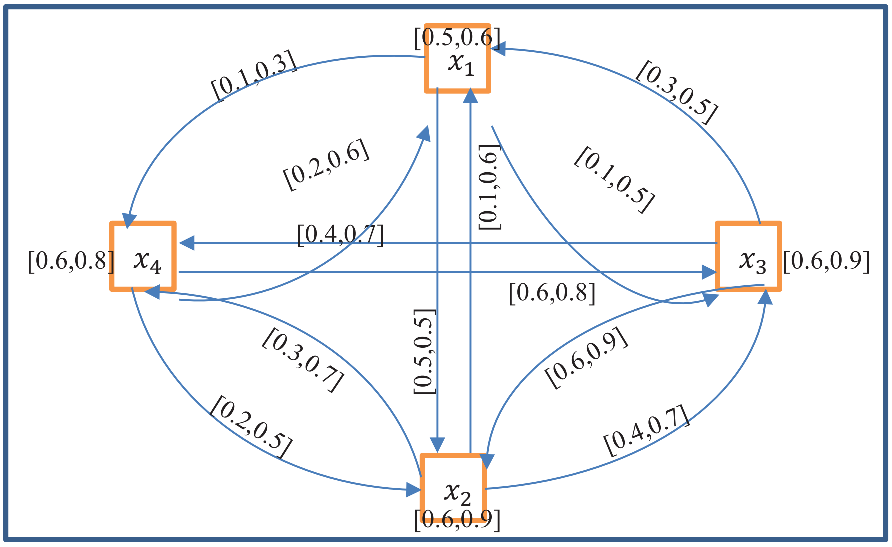

FIGURE 3. [IVF Directed Graph $\overrightarrow{\Gamma_2}_2$ (School 2)]

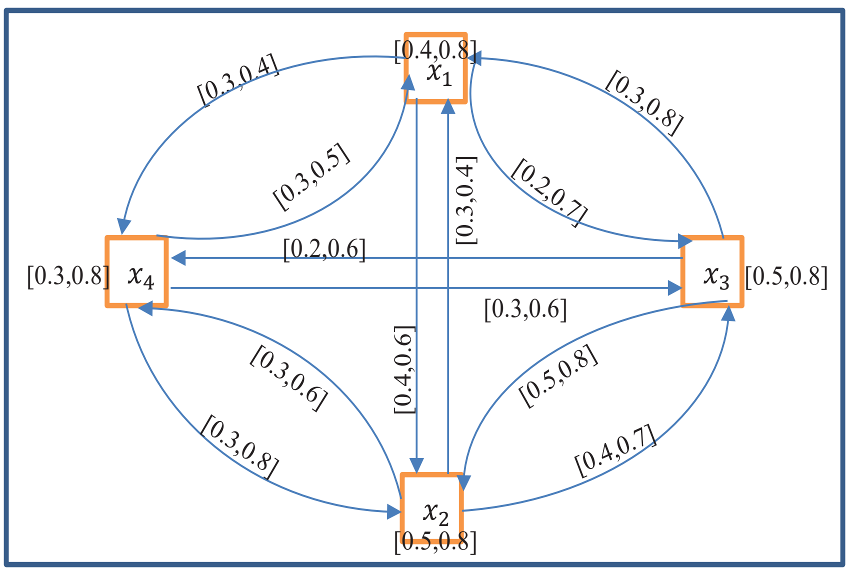

FIGURE 4. [IVF Directed Graph $\overrightarrow{\Gamma_3}$ (School 3)]

Conclusion

The IVFG can amplify flexibility and precision to model some problems better than an FG. Recently, graph energy has been employed in numerous fields. This is why we present the eigenvalues of the signless Laplacian matrix of the IVF-directed graph in this research. Moreover, we discuss the IVF-directed graph's spectrum, energy, and spectral radius using the corresponding signless Laplacian matrix.

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