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Properties for Certain Class of *p*- Valent Functions Related to Jackson's Operator

(Sifat bagi Kelas Fungsi Tertentu Valen - *p* yang Berkaitan dengan Pengoperasi Jackson)

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ABSTRACT

An inspiration from the fundamentals of (r,q) calculus to introduce an innovative subclass within the T(p) category of multivalent analytic functions, located within the confines of the open unit disk, is subjected to examination. The establishment of the subclass was achieved by employing Jackson's derivative operator to enhance the comprehension of these analytical functions. This article began by investigating and establishing adequate criteria that dictate the inclusion of functions within this recently introduced subclass. To achieve this, a comprehensive coefficient characterization to facilitate a deeper comprehension of the subclass's properties and behavior is derived. Further, various pertinent results that contribute to the broader understanding of the functions belonging to this subclass are explored. The findings and implications of these results are elucidated, underscoring the potential significance of this work in advancing the field of multivalent analytic functions and their applications. In conclusion, this paper broadens the scope of T(p) and sheds light on the distinct characteristics exhibited by the functions in this newly introduced subclass. This work sets the stage for further exploration and applications of (r,q) calculus and Jackson's derivative operator in the domain of multivalent analytic functions.

Keywords: p - valent function; quantum or (r,q)-calculus; (r,q)-derivative operator

ABSTRAK

Inspirasi daripada konsep asas kalkulus-(r,q) dengan memperkenalkan satu subkelas yang inovatif dalam kategori fungsi analisis multivalen T(p) tertakrif pada cakera unit terbuka akan dikaji. Pembinaan subkelas tercapai dengan menggunakan pengoperasi terbitan Jackson bagi meningkatkan kefahaman berkenaan fungsi analisis ini. Makalah ini dimulakan dengan mengkaji dan membina kriteria yang cukup bagi menyatakan rangkuman fungsi agar terkandung dalam subkelas yang diperkenalkan. Bagi mencapai hasrat tersebut, satu ciri pekali yang komprehensif diperoleh bagi memudahkan lagi kefahaman terhadap sifat dan kelakuan subkelas tersebut. Malah beberapa hasil penting yang menyumbang kepada kefahaman luas bagi subkelas fungsi ini dikaji. Keputusan dan implikasi hasil ini diperjelaskan dan menekankan potensi kepentingan kajian dalam memajukan bidang fungsi analisis multivalen dan penggunaannya. Kesimpulannya, makalah ini memperluaskan skop T(p) dan memberi pencerahan kepada ciri berbeza yang ditunjukkan oleh fungsi dalam subkelas baharu yang diperkenalkan. Kajian ini menyediakan ruang kepada penerokaan lanjutan dan penggunaan kalkulus-(r,q) dan pengoperasi terbitan Jackson dalam domain fungsi analisis multivalen.

Kata kunci: Fungsi valen p -; kuantum atau kalkulus- (r,q); pengoperasi terbitan- (r,q)

INTRODUCTION

This structure embodies the entirety of analytic functions is represented by the following formula

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n, \quad a_n \text{ is a complex number,}$$

which is defined within the open unit disk $\underline{\mathcal{U}} = \{z \in \mathbb{C}: |z| < 1\}$, where $\mathbb{C}$ represents the set of complex numbers, is indicated by $\widehat{\mathbb{A}}$. Let $\widehat{\mathbb{A}}(p)\ \ (p \in \mathbb{N} = \{1,2,3,\dots\})$ denote the set encompassing all analytic functions f. This category is characterized by functions possessing a series representation within the structure.

$$f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \text{ is complex number.}$$

That it is presented and called p- valent in $\mathcal{U}$ on the set of complex numbers $\mathbb{C}$. We note that $\widehat{\mathbb{A}}(1)=\widehat{\mathbb{A}}$. The category of functions encompassing all univalent functions within an open disk $\mathcal{U}$ is represented by $S(p)$, this forms a subset within $\widehat{\mathbb{A}}(p)$. Additionally, let $S_p^*(\alpha)$ and $C_p(\alpha)$ represent the categories of p-valent functions that are starlike of of order α and convex of order α, for $0 \leq \alpha < p$. Specifically, the categories $S_p^*(0)=S_p^*$ and $C_p(0)=C_p$ correspond to the well-known classes of starlike and convex p-valent functions within the open unit disk $\mathcal{U}$. After this, we assume that $\mathcal{T}(p)(p \in \mathbb{N}=\{1,2,3,\dots\})$ denote the subclass of $S(p)$ of analytic functions having the structure

$$f(z)=z^p-\sum_{n=p+1}^{\infty} a_n z^n, \qquad a_n>0 \tag{1.1}$$

This is defined on $\mathcal{U}=\{z\in\mathbb{C}:|z|<1\}$. A function $f \in \mathcal{T}(p)$ is known as a p- valent function with negative coefficients. Also, one can see that $S_{\mathcal{T},p}^*(\alpha)$ and $C_{\mathcal{T},p}(\alpha)$ for $0 \leq \alpha < p$, are p- valent functions, respectively, starlike of order α and convex of order α which is subclasses of $\mathcal{T}(p)$. The class $\mathcal{T}(1)=\mathcal{T}$ was derived by a study conducted by Silverman (1975), who studied and explored the subclasses of $\mathcal{T}(1)$ which is denoted by $S_{\mathcal{T},1}^*(\alpha)=S_{\mathcal{T}}^*(\alpha)$ and $C_{\mathcal{T},1}(\alpha)=C_{\mathcal{T}}(\alpha)$, for $0 \leq \alpha < 1$. These correspondingly denote starlike functions of order α and convex functions of order α.

To begin, we state mean known results and essential concepts of (r,q)-calculus. We let r,q be constants with $0<q<r\leq 1$ throughout the paper. We shall highlight some definitions and theorems of (r,q)-calculus. This is written and analyzed by these research papers (Kunt et al. 2018; Prabseang, Nonlaopon & Tariboon 2019). For $0<q<r\leq 1$, the Jackson`s (r,q) -derivative of a function $f\in\widehat{\mathbb{A}}(p)$ is given as follow

$$\mathcal{D}_{r,q}f(z) := \begin{cases} \dfrac{f(rz)-f(qz)}{(r-q)z}, & z\neq 0,\\ f'(0), & z=0. \end{cases} \tag{1.2}$$

From (1.2), we have

$$\mathcal{D}_{r,q}f(z)=[p]_{r,q}\, z^{p-1}+\sum_{n=p+1}^{\infty}[n]_{r,q}\; a_n\, z^{n-1}$$

$$(0<q<r\leq 1),$$

where $[p]_{r,q}=\frac{r^p-q^p}{r-q}$and $[n]_{r,q}=\frac{r^n-q^n}{r-q}$. As we note for $r = 1$, the Jackson (r,q)-derivative reduces to the Jackson q-derivative operator of the function f, $\mathcal{D}_q f(z)$, as explained in Alharayzeh (2021) and Jackson (1910, 1909). Note also that $\mathcal{D}_{1,q}f(z)\to f'(z)$ when $q\to 1$-, where f' is the classical derivative of the function f. Clearly for a function $g(z)=z^n$, we obtain.

$$\mathcal{D}_{r,q}g(z)=\mathcal{D}_{r,q}z^n=\frac{r^n-q^n}{r-q}z^{n-1}=[n]_{r,q}z^{n-1},$$

and

$$\lim_{q\to 1-}\mathcal{D}_{1,q}g(z)=\lim_{q\to 1-}\frac{1-q^n}{1-q}z^{n-1}=nz^{n-1}=g'(z),$$

where g'represents the standard derivative.

The field of applied science draws upon the principles of q-calculus to effectively tackle a diverse array of complex problems. This includes not only ordinary fractional calculus and quantum physics, but also extends to optimizing control systems, delving into intricate hypergeometric series, probing operator theories, and unraveling enigmas within complex analysis. Through the lens of q-calculus, these multifaceted challenges find resolution, demonstrating the remarkable versatility and applicability of this mathematical framework in complex analysis. Jackson (1909) initiated the application of q-calculus. Furthermore, fractional q-calculus operators are used to examine of specific class of functions that are analytic in $\mathcal{U}$ which studied by Kanas and Raducanu (2014). They have employed fractional q-calculus operators to analyze distinct categories of functions exhibiting analytic properties within the region $\mathcal{U}$. In tandem with the progress in q-calculus research, the variant of q-calculus reliant on two parameters (r,q)-integers has been introduced and garnered heightened attention over the recent decades. For example, by Chakrabarti and Jagannathan (1991) and Sadjang (2018). As of late, Tunç and Göv (2021) studied the (r,q)-derivative and (r,q)-integral on finite intervals. Besides, they concentrated on certain properties of (r,q)-calculus and (r,q)-associated with some important integral inequalities. The (r,q)-derivative have been considered and quickly created during this period by many creators.

Utilizing the previous defined (r,q)-calculus, certain new subclasses in the class $\widehat{\mathbb{A}}(p)$ in univalent geometric function theory is explored. A study conducted by Ismail, Merkes and Steyr (1990) who studied the first used the q-derivative operator $\mathcal{D}_q$, the study concentrated on the q-calculus comparable to the class S^* of starlike function in $\mathcal{U}$.

Now, let $\underline{\mathcal{M}}(A,B,t)$ be the subclass of $\widehat{\mathbb{A}}(1)$ consisting of functions $f \in \widehat{\mathbb{A}}(1)$ which satisfy the inequality $\left|\frac{f'(z)-1}{(A-B)t-B(f'(z)-1)}\right| < 1$, where $t \in \mathbb{C} \setminus \{0\}$ and $-1 \leq B < A \leq 1$ for all $z \in \underline{\mathcal{U}}$.

This class of functions was studied by Dix and Pal (1995). Further, we present some general subclass of analytic and multivalent functions that are related to (r,q)-derivative operator as follows.

Definition 1.1 For $t \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$, $k \geq 0$, $0 \leq c < 1$, $0 < q < r \leq 1$ and $p \in \mathbb{N} = \{1,2,3,...\}$, we let $\theta(t,A,B,c,k,r,q,p)$ composites of functions $f \in \mathcal{T}(p)$ that are satisfying the following condition

$$\operatorname{Re}\left(\frac{\left(\mathrm{D}_{r,q} f(z)\right)' - 1}{(A-B)t - B\left(\left(\mathrm{D}_{r,q} f(z)\right)' - 1\right)}\right) > k\left|\frac{\left(\mathrm{D}_{r,q} f(z)\right)' - 1}{(A-B)t - B\left(\left(\mathrm{D}_{r,q} f(z)\right)' - 1\right)} - 1\right| + c \tag{1.3}$$

The preliminary discovery pertains to the estimation of coefficients for function $f \in \Theta(t,A,B,c,k,r,q,p)$. We also verify the theorem of growth and distortion theorem. Moreover, we deduce the extreme points and the radius of starlikeness and convexity, for the include function f in the class $\theta(t,A,B,c,k,r,q,p)$. Primarily, let us focus on the coefficient inequalities for its importance, and the technique which studied in Aqlan, Jahangiri and Kulkarni (2004), Alharayzeh and Alzboon (2023), Alharayzeh and Ghanim (2022), and Alharayzeh and Darus (2010).

INEQUALITIES OF COEFFICIENTS

In this section, we are going to present the fundamental and necessary conditions for the function in the class $\theta(t,A,B,c,k,r,q,p)$.

Through this paper, we use the following notation.

$$\begin{aligned} \mu_n &= \left[(k+1)(B+1) - |B|(1-c)\right](n-1)[n]_{r,q}, \\ v_p &= \left[(k+1)(B+1) + |B|(1-c)\right](p-1)[p]_{r,q} \\ &\quad + (2+k-c)\left|(A-B)t+B\right| + k + 1, \end{aligned} \tag{2.1}$$

$$\mu_{p+1} = p\left[(k+1)(B+1) - |B|(1-c)\right][p+1]_{r,q}.$$

Where $-1 \leq B < A \leq 1, k \geq 0, 0 \leq c < 1, 0 < q < r \leq 1$ and $p \in \mathbb{N} = \{1,2,3,...\}$.

Our main results are depicted as shown below by the following.

Theorem 2.1 Assume , $t \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$, $k \geq 0$, $0 \leq c < 1$, $0 < q < r \leq 1$ and $p \in \mathbb{N} = \{1,2,3,...\}$. A function f given by (1.1) is in the class $\theta(t,A,B,c,k,r,q,p)$ then

$$\sum_{n=p+1}^{\infty} \mu_n a_n \leq v_p. \tag{2.2}$$

Where μ_n and v_p given by (2.1).

Proof. We have $f \in \theta(t,A,B,c,k,r,q,p)$ if and only if the condition (1.3) is satisfied.

Let

$$w = \frac{\left(\mathrm{D}_{r,q} f(z)\right)' - 1}{(A-B)t - B\left(\left(\mathrm{D}_{r,q} f(z)\right)' - 1\right)},$$

with, $Re(w) \geq k|w-1| + c$ if and only if $(k+1)|w-1| \leq 1-c$.

Now

$$(k+1)|w-1| = (k+1)\left|\frac{\sum_{n=p+1}^{\infty}[n]_{r,q}(n-1)a_n z^{n-2} - [p]_{r,q}(p-1)z^{p-2} + 1}{B[p]_{r,q}(p-1)z^{p-2} - B\sum_{n=p+1}^{\infty}[n]_{r,q}(n-1)a_n z^{n-2} - \left((A-B)t+B\right)} - 1\right| \leq 1-c,$$

is equivalent to

$$(k+1)\left|\frac{\sum_{n=p+1}^{\infty}(B+1)[n]_{r,q}(n-1)a_n z^{n-2} - (p-1)(B+1)[p]_{r,q} z^{p-2} + (A-B)t + B + 1}{B[p]_{r,q}(p-1)z^{p-2} - B\sum_{n=p+1}^{\infty}[n]_{r,q}(n-1)a_n z^{n-2} - \left((A-B)t+B\right)}\right| \leq 1-c. \tag{2.3}$$

The above inequality reduces to

$$(k+1)\frac{\left|\sum_{n=p+1}^{\infty}(B+1)[n]_{r,q}(n-1)a_n z^{n-2}\right| - \left|(p-1)(B+1)[p]_{r,q} z^{p-2}\right| - \left|(A-B)t+B+1\right|}{\left|B[p]_{r,q}(p-1)z^{p-2}\right| + \left|B\sum_{n=p+1}^{\infty}[n]_{r,q}(n-1)a_n z^{n-2}\right| + \left|(A-B)t+B\right|} \leq 1-c. \tag{2.4}$$

After that

$$(k+1)\frac{\sum_{n=p+1}^{\infty}(B+1)[n]_{r,q}(n-1)a_n - (p-1)(B+1)[p]_{r,q} - \left|(A-B)t+B+1\right|}{|B|[p]_{r,q}(p-1) + |B|\sum_{n=p+1}^{\infty}[n]_{r,q}(n-1)a_n + \left|(A-B)t+B\right|} \leq 1-c,$$

where, $|z|<1$. Now

$$(k+1)\left[\sum_{n=p+1}^{\infty}(B+1)[n]_{r,q}(n-1)a_n-(p-1)(B+1)[p]_{r,q}-|(A-B)t+B+1|\right]$$
$$\leq(1-c)\left[|B|[p]_{r,q}(p-1)+|B|\sum_{n=p+1}^{\infty}[n]_{r,q}(n-1)a_n+|(A-B)t+B|\right], \quad (2.5)$$

we have

$$\sum_{n=p+1}^{\infty}\left[(k+1)(1+B)-|B|(1-c)\right](n-1)[n]_{r,q}a_n$$
$$\leq \left[(k+1)(1+B)+|B|(1-c)\right][p]_{r,q}(p-1)+(k+1)|(A-B)t+B+1|$$
$$+(1-c)|(A-B)t+B|. \quad (2.6)$$

Thus

$$\sum_{n=p+1}^{\infty}\left[(k+1)(1+B)-|B|(1-c)\right](n-1)[n]_{r,q}a_n$$
$$\leq \left[(k+1)(1+B)+|B|(1-c)\right][p]_{r,q}(p-1)+(2+k-c)|(A-B)t$$
$$+B|+k+1,$$

which yield to (2.2).

Theorem 2.2 Let $t \in \mathbb{C} \setminus\{0\}$, $-1 \leq B < A \leq 1, k \geq 0, 0 \leq c < 1, 0 < q < r \leq 1$ and $p \in \mathbb{N} = \{1,2,3,...\}$. If the function f given by (1.1) is in the class $\theta(t,A,B,c,k,r,q,p)$ then

$$a_n \leq \frac{v_p}{\mu_n}, \qquad n = p+1, p+2, p+3,..., \quad (2.7)$$

where μ_n and v_p given by (2.1). Equality holds for the functions f given by,

$$f(z) = z^p - \frac{v_p z^n}{\mu_n}. \quad (2.8)$$

Proof. Since $f \in \theta$ (t,A,B,c,k,r,q,p), Theorem 2.1 holds. Now $\sum_{n=p+1}^{\infty}\mu_n a_n \leq v_p$ we have $a_n \leq \frac{v_p}{\mu_n}$. A function f given by (2.8) satisfies (2.7) and thus f is in $\theta(t, A, B, c, k, r, q, p)$. For this function the outcome is sharp.

GROWTH AND DISTORTION THEOREMS FOR THE SUBCLASS $\theta(t,A,B,c,k,r,q,p)$

The upcoming investigation will involve the examination of the growth and distortion for function f in the class $\theta(t,A,B,c,k,r,q,p)$ as elucidated by the subsequent theorems.

Theorem 3.1 Let $t \in \mathbb{C} \setminus\{0\}$, $-1 \leq B < A \leq 1, k \geq 0, 0 \leq c < 1, 0 < q < r \leq 1$ and $p \in \mathbb{N} = \{1,2,3,...\}$. If the function f as expressed in (1.1) and it is a member of the $\theta(t,A,B,c,k,r,q,p)$ then for $0 < |z| = l < 1$, we get

$$l^p - \frac{v_p}{\mu_{p+1}}l^{p+1} \leq |f(z)| \leq l^p + \frac{v_p}{\mu_{p+1}}l^{p+1}, \quad (3.1)$$

equality holds for the function,

$$f(z) = z^p - \frac{v_p}{\mu_{p+1}}z^{p+1}, \quad (z = \pm l, \pm il),$$

where v_p and μ_{p+1} can be found by (2.1).

Proof. We start from the right-hand side inequality in (3.1). Since $f \in \theta$ (t,A,B,c,k,r,q,p) by Theorem 2.1 we have, $\sum_{n=p+1}^{\infty}\mu_n a_n \leq v_p$. Now

$$\mu_{p+1}\sum_{n=p+1}^{\infty}a_n \leq \sum_{n=p+1}^{\infty}\mu_{p+1}a_n \leq \sum_{n=p+1}^{\infty}\mu_n a_n \leq v_p.$$

And therefore

$$\sum_{n=p+1}^{\infty}a_n \leq \frac{v_p}{\mu_{p+1}}, \quad (3.2)$$

since $f(z)=z^p-\sum_{n=p+1}^{\infty}a_n z^n$, $a_n > 0$ we have,

$$|f(z)| = \left|z^p - \sum_{n=p+1}^{\infty}a_n z^n\right| \leq |z|^p + |z|^{p+1}\sum_{n=p+1}^{\infty}$$
$$a_n|z|^{n-(p+1)} \leq l^p + l^{p+1}\sum_{n=p+1}^{\infty}a_n.$$

From (3.2), yields the right-hand side inequality of (3.1). And for the other inequality

$$l^p - l^{p+1}\sum_{n=p+1}^{\infty}a_n \leq |z|^p - |z|^{p+1}\sum_{n=p+1}^{\infty}a_n|z|^{n-(p+1)} \leq \left|z^p\right.$$
$$\left.-\sum_{n=p+1}^{\infty}a_n z^n\right| = |f(z)|, \qquad |z|<1.$$

From (3.2), yields the left-hand side inequality of (3.1).

Theorem 3.2 If the function f given by (1.1) is in the class $\theta(t,A,B,c,k,r,q,p)$ for $0 < |z| = l < 1$, then we have

$$pl^{p-1}-\frac{(p+1)v_p}{\mu_{p+1}}l^p \le |f'(z)| \le pl^{p-1}+\frac{(p+1)v_p}{\mu_{p+1}}l^p. \quad (3.3)$$

Equality holds for the function f given by

$$f(z)=z^p-\frac{v_p}{\mu_{p+1}}z^{p+1}, \qquad (z=\pm l,\ \pm il),$$

where v_p and μ_{p+1} can be found by (2.1).

Proof. Since $f \in \theta\,(t,A,B,c,k,r,q,p)$ by Theorem 2.1 we have $\sum_{n=p+1}^{\infty}\mu_n a_n \le v_p$.

Now, $\mu_{p+1}\sum_{n=p+1}^{\infty} na_n \le (p+1)\sum_{n=p+1}^{\infty}\mu_n a_n \le (p+1)v_p$, hence

$$\sum_{n=p+1}^{\infty} na_n \le \frac{(p+1)v_p}{\mu_{p+1}}. \quad (3.4)$$

Since $f'(z)=pz^{p-1}-\sum_{n=p+1}^{\infty}na_n z^{n-1}$. Then, we have

$$p|z|^{p-1}-|z|^p\sum_{n=p+1}^{\infty}na_n|z|^{n-1-p} \le |f'(z)| \le p|z|^{p-1}$$

$$+|z|^p\sum_{n=p+1}^{\infty}na_n|z|^{n-1-p}, \quad \text{where} \quad |z|<1.$$

By applying the inequality (3.4), we extracted Theorem 3.2, and this will complete the proof.

Theorem 3.3 If the function f given by (1.1) is in the class $\theta\,(t,A,B,c,k,r,q,p)$ it follows that f exhibits a starlike of order δ, where

$$\delta = 1-\frac{p\,v_p}{-v_p+\mu_{p+1}}.$$

The result is sharp with

$$f(z)=z^p-\frac{v_p}{\mu_{p+1}}z^{p+1},$$

where v_p and μ_{p+1} can be found from (2.1).

Proof. Demonstrating (2.1) is sufficient to establish the implication.

$$\sum_{n=p+1}^{\infty}a_n(n-\delta)\le 1-\delta. \quad (3.5)$$

That is,

$$\frac{n-\delta}{1-\delta}\le\frac{\mu_n}{v_p}, \qquad n\ge p+1. \quad (3.6)$$

The above inequality is equivalent to.

$$\delta \le 1-\frac{(n-1)v_p}{-v_p+\mu_n}=\psi(n),$$

where $n \ge p+1$. Since $\psi(n) \ge \psi(p+1)$, (3.6) holds true for any $t \in \mathbb{C}\setminus\{0\}$, $-1 \le B < A \le 1$, $k \ge 0$, $0 \le c < 1$, $0 < q < r \le 1$ and $p \in \mathbb{N} = \{1,2,3,...\}$. This completes the proof of Theorem 3.3.

EXTREME POINTS WITHIN CLASS $\theta(t,A,B,c,k,r,q,p)$

The extreme points of the class $\theta(t,A,B,c,k,r,q,p)$ given by the following theorem.

Theorem 4.1 Let $f_p(z)=z^p$ and $f_n(z)=z^p-\frac{v_p}{\mu_n}z^n$, $n = p+1, p+1, p+3, ...$, where μ_n and v_p given by (2.1). If $f \in \theta\,(t,A,B,c,k,r,q,p)$ then f can be represented in the form

$$f(z)=\sum_{n=p}^{\infty}y_n f_n(z) \quad (4.1)$$

where $y_n \ge 0$ and $\sum_{n=p}^{\infty}y_n=1$

Proof. Assume the function $f \in \theta\,(t,A,B,c,k,r,q,p)$ our target is to get (4.1). Now from (4.1) it easily seen that

$$f(z)=\sum_{n=p}^{\infty}y_n f_n(z)=y_p f_p(z)+\sum_{n=p+1}^{\infty}y_n f_n(z)$$

$$=y_p f_p(z)+\sum_{n=p+1}^{\infty}y_n\left[z^p-\frac{v_p}{\mu_n}z^n\right],$$

$$=y_p f_p(z)+\sum_{n=p+1}^{\infty}y_n z^p-\sum_{n=p+1}^{\infty}y_n\frac{v_p}{\mu_n}z^n$$

$$=\sum_{n=p}^{\infty}y_n z^p-\sum_{n=p+1}^{\infty}y_n\frac{v_p}{\mu_n}z^n,$$

$$=z^p-\sum_{n=p+1}^{\infty}y_n\frac{v_p}{\mu_n}z^n.$$

Now it is suffices to show that $a_n = y_n \frac{v_p}{\mu_n}$, $n \ge p+1$.

Now, we have $f \in \theta$ (t,A,B,c,k,r,q,p) then by previous Theorem 2.2.

$$a_n \le \frac{v_p}{\mu_n}, \quad n \ge p+1 \text{ that is } \frac{\mu_n}{v_p} a_n \le 1.$$

Now $y_n \ge 0$ and $\sum_{n=p}^{\infty} y_n = 1$ then we see $\sum_{n=p}^{\infty} y_n = y_p + \sum_{n=p+1}^{\infty} y_n = 1$ hence $\sum_{n=p+1}^{\infty} y_n = 1 - y_p \le 1$.

And since $\sum_{n=p+1}^{\infty} y_n \le 1$ we get $y_n \le 1$ for each $n = p+1, p+2, p+3, \ldots$ and $p = 1,2,3,\ldots$.

Setting $y_n = \frac{\mu_n}{v_p} a_n$, thus the desired result is that $a_n = y_n \frac{v_p}{\mu_n}$. This completes the proof of the theorem.

Corollary 4.2 The extreme point of the class θ (t,A,B,c,k,r,q,p) is the function.

$$f_p(z) = z^p,$$

and

$$f_n(z) = z^p - \frac{v_p}{\mu_n} z^n, \quad n = p+1, p+2, p+3, \ldots,$$

where μ_n and v_p given by (2.1).

Theorem 4.3 The class θ (t,A,B,c,k,r,q,p) is closed with respect to convex linear combinations.

Proof. Suppose that the functions $f_1(z)$ and $f_2(z)$ defined by

$$f_i(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,i} z^n, \quad (i = 1,2;\ z \in \underline{u}) \tag{4.2}$$

That are in the class θ (t,A,B,c,k,r,q,p). By setting

$$f(z) = cf_1(z) + (1-c)f_2(z) \qquad (0 \le c \le 1),$$

we find from (4.2) the following

$$f(z) = z^p - \sum_{n=p+1}^{\infty} \left(ca_{n,1} + (1-c)a_{n,2}\right) z^n, \ (0 \le c \le 1;\ z \in \underline{u}).$$

In view of Theorem 2.1, we have

$$\sum_{n=p+1}^{\infty} \mu_n (ca_{n,1} + (1-c)a_{n,2}) = c \sum_{n=p+1}^{\infty} \mu_n a_{n,1} +$$

$$(1-c) \sum_{n=p+1}^{\infty} \mu_n a_{n,2} \le c v_p + (1-c) v_p = v_p$$

which shows that $f \in \theta$ (t,A,B,c,k,r,q,p). Hence the theorem is based on.

Conclusively, within this paper, we establish the fulfillment of the radius conditions for both starlikeness and convexity concerning the function f belonging to the class θ (t,A,B,c,k,r,q,p) as demonstrated.

RADIUS OF STARLIKENESS AND CONVEXITY

The radius of starlikeness and convexity for the function f in the class θ (t,A,B,c,k,r,q,p) will also be considered.

Theorem 5.1 If the function f given by (1.1) is in the class θ (t,A,B,c,k,r,q,p), then f is starlike of order δ $(0 \le \delta < p)$, in the disk $|z| < R$, where

$$R = \inf\left[\frac{\mu_n}{v_p} \times \left(\frac{p-\delta}{n-\delta}\right)\right]^{\frac{1}{n-p}}, \quad n = p+1, p+2, p+3, \ldots, \tag{5.1}$$

where μ_n and v_p given by (2.1).

Proof. Here (5.1) implies $v_p(n-\delta)|z|^{n-P} \le \mu_n(p-\delta)$. It suffices to show that $\left|\frac{zf'(z)}{f(z)} - p\right| \le p - \delta$, for $|z| < R$, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le \frac{\sum_{n=p+1}^{\infty} (n-p) a_n |z|^{n-p}}{1 - \sum_{n=p+1}^{\infty} a_n |z|^{n-p}}. \tag{5.2}$$

By aid of (2.8), we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le \frac{\sum_{n=p+1}^{\infty} \frac{v_p (n-p)|z|^{n-p}}{\mu_n}}{1 - \sum_{n=p+1}^{\infty} \frac{v_p |z|^{n-p}}{\mu_n}}.$$

The last expression is confined by $p-\delta$ if

$$\sum_{n=p+1}^{\infty} \frac{v_p(n-p)|z|^{n-p}}{\mu_n} \le \left[1 - \sum_{n=p+1}^{\infty} \frac{v_p |z|^{n-p}}{\mu_n}\right](p-\delta),$$

and it follows that

$$|z|^{n-p} \le \frac{\mu_n}{v_p}\left(\frac{p-\delta}{n-\delta}\right), \quad n \ge p+1,$$

which is equivalent to our condition (5.1) of the theorem.

Theorem 5.2 If the function f given by (1.1) is in the class $\theta\ (t,A,B,c,k,r,q,p)$ then f is convex of order ε $(0 \le \varepsilon < p)$, in the disk $|z| < w$ where

$$w = \inf\left[\frac{\mu_n}{v_p} \times \left(\frac{p(p-\varepsilon)}{n(n-\varepsilon)}\right)\right]^{\frac{1}{n-p}}, \qquad n = p+1, p+2, p+3, ...,$$

where μ_n and v_p given by (2.1).

Proof. By employing the identical technique as employed in the proof of Theorem 5.1, we can demonstrate that.

$$\left|\frac{zf''(z)}{f'(z)} - (p-1)\right| \le p - \varepsilon, \qquad \text{for} \qquad |z| \le w,$$

with the aid of (2.8). Thus, we have the assertion of Theorem 5.2.

Conclusion

This paper has successfully explained the search for a novel subclass of multivalent analytic functions on the open unit disc, employing the (r,q) derivative operator as a defining criterion. The primary focus of this investigation was to establish coefficient characterization within specific classes of functions. The results obtained through this approach have proven to be of great interest and significance, offering valuable insights into estimation of coefficients, distortion and growth theorems, extreme points, convexity of functions, and starlikeness radius. Moreover, this study showed the possibility of deriving extended classes of multivalent analytic functions using the (r,q) derivative operator. These discoveries present a novel opportunity to extend the horizons of subsequent investigations and to attain a more profound comprehension of the attributes exhibited by p-valent functions in conjunction with Jackson's operator. Ultimately, the systematic examination of this specialized function category alongside the application of the (r,q) derivative operator has significantly enhanced our cognitive grasp of the inherent traits of multivalent analytic functions. Furthermore, these accomplishments have established a fundamental framework upon which forthcoming inquiries within this realm can be constructed. The results presented herein contribute significantly to the existing body of knowledge and present promising opportunities for advancing the field of p-valent functions in connection with Jackson's operator.

We remark that some extended classes of multivalent analytic functions can be derived with (r,q) derivative operator and studied their coefficients characterization. Also these results can be extended to study problems related to the Fekete-Szego theorem (Harayzeh & Darus 2011) as a future research.

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