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Modeling of Gross Domestic Product with Foreign Direct Investment using Lotka-Volterra Equations

(Pemodelan Keluaran Dalam Negara Kasar dengan Pelaburan Langsung Asing menggunakan Persamaan Lotka-Volterra)

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ABSTRACT

This paper investigates the dynamic interaction between Gross Domestic Product (GDP) and Foreign Direct Investment (FDI) using two distinct numerical methods: The Fourth-Order Runge-Kutta (RK4) method on Lotka-Volterra model and a family of Least-Squares (LS) methods. The study aims to provide a comparative analysis of these methods in terms of their accuracy, efficiency, and applicability in modeling the complex relationship between GDP and FDI. The RK4 method is employed to model the dynamic system governing the interaction between GDP and FDI. This method is chosen for its robustness in handling non-linear systems and its ability to provide precise numerical solutions with minimal computational error. On the other hand, the least squares method provides a static approximation by fitting a linear or nonlinear relationship between GDP and FDI. The paper conducts simulations using real-world data on GDP and FDI from Malaysia spanning the years 2009 to 2020. The results obtained from both methods are compared to assess their performance. The RK4 method on Lotka-Volterra model demonstrates superior accuracy in capturing the dynamic behavior of the GDP-FDI interaction, particularly in scenarios involving rapid changes or non-linear dynamics.

Keywords: Dynamic interaction; Foreign Direct Investment (FDI); Fourth-Order Runge-Kutta (RK4); Gross Domestic Product (GDP); Lotka-Volterra (LV)

ABSTRAK

Kertas ini mengkaji interaksi dinamik antara Keluaran Dalam Negara Kasar (KDNK) dan Pelaburan Langsung Asing (PLA) dengan menggunakan dua kaedah berangka yang berbeza: Kaedah Runge-Kutta Tertib Keempat (RK4) pada model Lotka-Volterra dan satu keluarga kaedah Kuasa Dua Terkecil. Kajian ini bertujuan untuk memberikan analisis perbandingan antara kaedah ini dari segi ketepatan, kecekapan dan kebolehgunaan dalam memodelkan hubungan kompleks antara KDNK dan PLA. Kaedah RK4 digunakan untuk memodelkan sistem dinamik yang mengawal interaksi antara KDNK dan PLA. Kaedah ini dipilih kerana keteguhannya dalam mengendalikan sistem bukan linear dan keupayaannya memberikan penyelesaian berangka yang tepat dengan kesilapan pengiraan yang minimum. Sebaliknya, kaedah kuasa dua terkecil memberikan suatu anggaran statik dengan memadankan hubungan linear atau tak linear antara KDNK dan PLA. Kertas ini menjalankan simulasi menggunakan data dunia sebenar KDNK dan PLA dari Malaysia bagi tempoh 2009 hingga 2020. Hasil yang diperoleh daripada kedua-dua kaedah ini dibandingkan untuk menilai prestasi mereka. Kaedah RK4 pada model Lotka-Volterra menunjukkan ketepatan yang lebih tinggi dalam mencerap tingkah laku dinamik interaksi KDNK-PLA, terutamanya dalam senario yang melibatkan perubahan pantas atau dinamik bukan linear.

Kata kunci: Interaksi dinamik; Keluaran Dalam Negara Kasar (KDNK); Lotka-Volterra (LV); Pelaburan Langsung Asing (PLA); Runge-Kutta Tertib Keempat (RK4)

INTRODUCTION

The relationship between Gross Domestic Product (GDP) and Foreign Direct Investment (FDI) (Joo & Shawl 2023) is a cornerstone of economic theory and policy-making, as it significantly influences a country's economic growth and development. GDP, a measure of the economic output and health of a nation, is often seen as a driver of FDI, attracting foreign capital by signaling a stable and prosperous economy. Conversely, FDI is frequently viewed as a catalyst for GDP growth, bringing in not just capital, but also technology, skills, and access to global markets. Understanding the dynamic interaction between GDP and FDI is crucial for developing effective economic policies that foster sustainable growth (Mwakabungu & Kauangal 2023; Mwinuka & Mwangoka 2023).

Examining the relationship between GDP and FDI is crucial for policymakers focused on promoting economic growth (Joo & Shawl 2023). Many studies emphasize FDI's role as a driver of GDP growth, particularly in developing economies (Borensztein, De Gregorio & Lee 1998; De Mello 1997; Joo & Shawl 2023), often employing linear LS models to measure this impact. For instance, research by Alfaro et al. (2004) and Borensztein, De Gregorio and Lee (1998) demonstrates a positive correlation between higher GDP growth rates and increased FDI inflows. These findings suggest that economies with strong growth prospects are more attractive to foreign investors due to the potential for higher returns. Additionally, Agosin and Mayer (2000) and De Mello (1997) argue that FDI stimulates economic development by bringing in capital, transferring technology, and providing managerial expertise, all of which enhance productivity and innovation. Their research indicates that FDI boosts GDP by encouraging domestic investment and employment.

In the ASEAN region, Sijabat (2023) identifies a bidirectional causal relationship between FDI and GDP, particularly in the short term, underscoring the reciprocal nature of this interaction. Conversely, a nation's GDP can also influence FDI inflows, as economic stability and growth prospects either attract or deter foreign investment. While the positive effects of FDI on economic growth are well-documented, there is also evidence that FDI can have negative consequences in certain situations. For example, Nguyen (2022) suggests that excessive and poorly regulated FDI inflows may hinder economic growth in the host country. Similarly, studies by Görg and Greenaway (2004), Gui-Diby (2014), and Sumner (2005) highlight the varying effects of FDI on development. Notably, Gui-Diby (2014) found that FDI had a negative impact on economic growth in 50 African countries between 1980 and 1994. However, this trend reversed between 1995 and 2009, with FDI beginning to positively contribute to economic growth. These findings highlight the complex and context-dependent nature of the GDP-FDI relationship, illustrating that the effects of FDI on growth are multifaceted.

Linear LS is widely used across fields such as economics, finance, and social sciences due to its simplicity and ease of interpretation (Wooldridge 2016). In medicine, it is valuable for predicting outcomes based on factors like age and treatment dosage (Montgomery, Peck & Vining 2021). Its transparency makes linear regression a powerful tool for both understanding data relationships and making predictions. However, linear LS has limitations, particularly in dealing with real-world data that often exhibit nonlinear relationships, limiting its applicability in such cases (Fox 2016; Wang et al. 2018). The model often relies on assumptions of linearity and equilibrium, which may not fully capture the complex, non-linear, and dynamic nature of the interaction between GDP and FDI. Additionally, linear LS is sensitive to outliers, which can distort the results significantly (Wooldridge 2016). Another challenge arises when independent variables are highly correlated, complicating the interpretation of coefficients (Montgomery, Peck & Vining 2021).

In contrast, cubic and quadratic LS are used to model relationships where variables interact in a nonlinear manner curve (Helene, Mariano & Guimarães-Filho 2016; Sima, Van Huffel & Golub 2004; Wang et al. 2018). Unlike linear LS, where the parameters are linear with respect to the independent variables, cubic LS captures nonlinear dynamics, making it better suited for complex relationships. For example, in biology, nonlinear LS is used to model growth curves and enzyme kinetics (Ratkowsky 1983), while in finance, it helps capture the intricate behaviour of asset prices and interest rates (Campbell, Lo & MacKinlay 1997). By incorporating nonlinear terms, such as polynomial terms, cubic LS expands the capacity of linear models to capture more complex relationships without fully embracing the complexity of a nonlinear model (Montgomery, Peck & Vining 2021). However, nonlinear LS requires careful model selection and validation to avoid issues like overfitting and convergence problems (Seber & Wild 2003). However, all of these LS models represent the relation of GDP and FDI statically.

In Malaysia, previous research has highlighted a significant relationship between FDI and economic growth (Ang 2008; Lean & Tan 2011; Mohamed, Jit Sing & Liew 2013). However, these studies predominantly utilized linear models, which may have missed the bidirectional causality between GDP and FDI. To address this limitation, this study models the Malaysian dataset using a Lotka-Volterra-like model and simulates it with the RK4 method (Hasan, Karim & Sulaiman 2015; Hussain, Ismail & Senu 2016), providing a more nuanced understanding of the GDP-FDI relationship. The RK4 method, renowned for its accuracy in solving ordinary differential equations (ODEs), is particularly well-suited for modeling non-linear dynamic systems.

The dynamic interaction between GDP and FDI can be effectively modelled using a Lotka-Volterra-like approach (Hasan, Othman & Idrus 2022, 2019), which is traditionally used to describe predator-prey systems in ecology. Hasan, Othman and Idrus (2019) apply the Adam-BDF function in lsoda algorithm, while Hasan, Othman and Idrus (2022) apply nonstandard finite difference method with trimean nonlinear approximation. In contrast, Wu and Liu (2013) apply the grey Lotka-Volterra model to analyse the interaction between GDP and FDI. In this context, GDP and FDI can be thought of as interacting populations, where GDP influences FDI and vice versa. This type of modelling allows for the capture of complex, nonlinear interactions that linear models might fail to address. We compare the effectiveness of this approach with various LS in analysing the GDP-FDI relationship. By applying the RK4 method and LS methods to real-world Malaysian economic data, the study seeks to identify the strengths and limitations of each approach. The findings are expected to enhance the economic modeling literature by providing insights into the most suitable methods for capturing the complex dynamics between GDP and FDI, thereby offering valuable guidance for policymakers and economists.

MATERIALS AND METHODS

The Lotka-Volterra model, originally developed to describe predator-prey dynamics in biological systems, can be adapted to model the interaction between two macroeconomic variables, such as GDP and FDI. This kind of approach assumes that the two variables influence each other in a way analogous to predator-prey or competitive interactions. We propose a Lotka-Volterra like model for FDI and GDP.

Step 1: Idea on the interacting equations

The general form of the Lotka-Volterra equations for two interacting macroeconomic factors includes two equations. The first equation has two components; i.e., Growth rate of GDP in the absence of FDI and the impact of FDI to GDP. While for the second equation also have two components; i.e., Decay rate of FDI in the absence of GDP and the positive impact of GDP on FDI. We can write these ideas in general forms as in (1):

$$\left.\begin{aligned} \frac{dGDP}{dt} &= \alpha GDP - \beta GDP * FDI \\ \frac{dGDP}{dt} &= \delta FDI - GDP * \gamma FDI \end{aligned}\right\} \quad (1)$$

where *α, β, γ* and *δ* are parameters that describe the interaction between the factors.

The equations will describe the rates of GDP and FDI changes over time, considering their interaction. To be realistic to the factors considered, we define α as growth rate of GDP in the absence of FDI. This represents the natural growth of the economy without foreign investment. β as the negative impact of FDI on GDP. We view FDI as a competitor or drain analogous to a 'predator' in species. The sign could also be positive if FDI have positive impact on GDP if the relationship is symbiotic. γ as decay rate of FDI in the absence of GDP. This represents how FDI decreases if the economy (GDP) is not growing or is stagnant and δ as the positive impact of GDP on FDI, which reflects the idea that a growing economy attracts more foreign investment.

Step 2 Refine the GDP and FDI rates equations

The first terms in the differential equations for rates of change in GDP represent different dynamics that affect the growth or decline of GDP over time. This term can be refined in more detail. In real world, the GDP can be incline and sometimes decline. The negative term could represent factors like depreciation, natural resource depletion, or other forces that cause GDP to decrease proportionally as the economy grows. This term can also capture aspects like diminishing returns, where each additional unit of GDP becomes harder to maintain. This term alone would suggest that GDP would eventually decline to zero if there were no other factors counteracting this decay. Therefore, we refine the growth of GDP as $GDP(\alpha_1 - \alpha_2 \text{GDP})$.

The second terms in the differential equations for rate of change in FDI would also experience similar case to GDP which sometime increase and, in another situation, decrease. In real situation, the declining of FDI could represent various reason such as Capital Repatriation: Foreign investors might withdraw their investments as profits are repatriated back to the home country, reducing the level of FDI in the host country. Political or Economic Risks: As FDI increases, the host country might face political or economic challenges that make it less attractive to foreign investors, leading to a withdrawal or reduction in new investments. Saturation Effects: In certain sectors, as more FDI flows in, opportunities for profitable investments might diminish, leading to a natural slowdown in FDI inflows. This term alone suggests that FDI will naturally decline over time unless counteracted by other positive factors. This could be seen as a baseline tendency of FDI to decrease due to natural market dynamics, competition, or external risks. Thus, we refine the growth of FDI as $FDI\,(\gamma_1 - \gamma_2 \text{FDI})$.

Therefore, our new refined Lotka-Volterra model for GDP and FDI are rewritten as in (2):

$$\left.\begin{aligned} \frac{dGDP}{dt} &= GDP\,(a_1 - a_2 GDP) - \beta GDP * FDI \\ \frac{dGDP}{dt} &= \delta FDI * GDP - FDI\,(\gamma_1 - \gamma_2 FDI) \end{aligned}\right\} \quad (2)$$

Step 3 Explain the Model Components

The given system of differential equations represents a modified Lotka-Volterra model to describe the interaction between GDP and FDI. Each equation describes the rate of change of GDP and FDI over time, incorporating both internal growth/decay and interaction effects between the two variables. The first equation explanation is as follows.

$$\frac{dGDP}{dt} = \text{GDP}(\alpha_1 - \alpha_2 \text{GDP}) - \beta GDP * FDI$$

The first term, $GDP(\alpha_1 - \alpha_2 \text{ GDP})$ represents the intrinsic growth rate of GDP. It assumes that GDP grows at a rate proportional to its current size, α_1, which can be interpreted as the natural growth rate (if α_1 is positive) of the economy in the absence of any limiting factors, but is represents natural decay or diminishing returns that slow down GDP growth if α_1 is negative. It's proportional to the current level of GDP, indicating that as GDP increases, these negative forces become stronger. This could be because of economic growth driven by factors like investment, consumption, and productivity, independent of the interaction with FDI. The α_2 on the other hand represent self-limiting factors (if α_2 is positive) or crowding effect on GDP that limit or decaying the growth of GDP (representing constraints such as resource limitations, diminishing returns, or economic saturation), but represent positive feedback or compounding effects (if α_2 is negative) that accelerate GDP growth as it increases. It suggests that beyond a certain point, growth becomes self-reinforcing due to factors like economies of scale or network effects.

The term $-\beta GDP * FDI$ captures the interaction between GDP and FDI. The negative sign suggests that FDI has a dampening effect on GDP growth. This could represent a scenario where foreign investments may crowd out domestic investments, lead to profit repatriation, or impose other economic costs on the domestic economy. The constant, β represents the intensity of the interaction between GDP and FDI. A larger value of β indicates a stronger negative impact of FDI on GDP growth. The term reflects the idea that while FDI can contribute to economic growth, it can also have negative side effects, such as increased competition for domestic firms, dependency on foreign capital, or external influences on the economy.
The second equation explanation is as follows.

$$\frac{dFDI}{dt} = \delta FDI * GDP - FDI\,(\gamma_1 - \gamma_2 FDI)$$

The term $\delta FDI * GDP$ represents the positive feedback from GDP to FDI. The higher the GDP, the more attractive the economy becomes to foreign investors, leading to an increase in FDI. The positive constant, δ indicates how strongly GDP influences FDI. A larger value δ implies a stronger positive impact of GDP on FDI inflows. This term reflects the idea that a growing economy attracts more foreign investment due to factors such as higher returns, better opportunities, and increased market stability.

The term $-\gamma_1 FDI$ represents the natural decay or reduction in FDI over time, in the absence of economic growth. It reflects the idea that FDI would naturally diminish without sustained economic performance. This constant, γ_1 t represents the rate of FDI decay due to factors such as capital outflows, profit repatriation, or economic instability. This term ensures that FDI does not grow indefinitely and accounts for the natural tendency of foreign investments to withdraw or diminish over time if the economic environment becomes less favourable. The term, $\gamma_2 FDI^2$ introduces a self-limiting effect on FDI. As FDI increases, this term reduces the growth rate, representing constraints such as market saturation, diminishing returns, or capital constraints. This positive constant, γ_2 represents the strength of the self-limiting effect on FDI. It prevents FDI from growing uncontrollably and ensures the stability of the system.

This term accounts for the natural limits on how much foreign investment an economy can absorb before diminishing returns or negative effects set in. For instance, excessive FDI might lead to market saturation, over-dependence on foreign capital, or increased vulnerability to external shocks.

Step 4 Parameters optimisation

In this paper, we apply the Nelder-Mead Simplex method (Hasan, Othman & Idrus 2019) to gather optimized values for all model parameters (α_1 , α_2 β, δ ,γ_1 and γ_2) by iteratively exploring the parameter space and minimizing the objective function. This derivative-free optimization technique is particularly suitable for nonlinear problems and enables efficient convergence to the best-fit parameter set that aligns with the observed data. The parameters gathered are $\alpha_1 = -12.65$, $\alpha_2 = -3.36$, $\beta = 2.26$, γ_1 44.65, $\gamma_2 = -6.17$, $\delta = 10.83$. However, this coefficient parameters are different from Hasan, Othman and Idrus (2019) may be because of their paper utilized data from 2009-2018. Therefore, our GDP and FDI Lotka-Volterra for Malaysian Dataset from 2009-2020 are given in (3):

$$\left.\begin{aligned} \frac{dGDP}{dt} &= -12.65GDP + 3.36GDP^2 - 2.26GDP * FDI \\ \frac{dFDI}{dt} &= -44.65FDI - 6.17FDI^2 + 10.83GDP * FDI \end{aligned}\right\} \quad (3)$$

Both differential model in Equation (3) explain the complex interaction for GDP and FDI. The first differential equation $\frac{dGDP}{dt} = -12.65\ GDP + 3.36\ GDP^2 - 2.26\ GDP * FDI$ describes how the Gross Domestic Product (GDP) changes over time, considering both the intrinsic factors affecting GDP and the interaction between GDP and Foreign Direct Investment (FDI). The terms $-12.65GDP + 3.36GDP^2$ represent the intrinsic GDP Growth Component.

The term $-12.65GDP$ represents a negative influence on GDP, which could be due to factors such as economic decay, resource depletion, or other factors that naturally slow down GDP growth. While the term $3.36GDP^2$ represents a quadratic influence on GDP, suggesting that as GDP increases, there is a positive feedback loop that accelerates GDP growth. This might reflect factors such as economies of scale, increasing returns on investment, or enhanced productivity. The combination of these terms can be understood as the internal dynamics of GDP growth. For lower GDP levels, the negative term -12.65 may dominate, leading to a decrease in GDP. As GDP increases, the positive quadratic term 3.36 GDP becomes more significant, potentially overcoming the negative influence and leading to an increase in GDP. The term $-2.26GDP * FDI$ represents the interaction between GDP and FDI, with a negative coefficient indicating that this interaction reduces the rate of GDP growth. This could imply that under certain conditions, higher levels of FDI might create economic dependencies or resource reallocations that detract from domestic GDP growth. For instance, large inflows of FDI might lead to capital outflows or profits being repatriated abroad, which could dampen the domestic GDP.

In essence, the equation $\frac{dGDP}{dt}$ = –12.65 *GDP* + 3.36 GDP^2 – 2.26 *GDP* * *FDI* captures the complex relationship between GDP growth, its internal dynamics, and the influence of FDI. It suggests that GDP growth is influenced by both its current level and the level of FDI, leading to potential positive or negative growth depending on these factors. The negative interaction term underscores the importance of managing FDI to ensure it supports, rather than hinders, domestic economic growth.

The second differential equation $\frac{dFDI}{dt}$= –44.65 *FDI* – 6.17FDI^2 + 10.83 *GDP* * *FDI* describes how FDI changes over time, considering both the internal factors affecting FDI and its interaction with GDP. The intrinsic FDI growth component for this equation is –44.65*FDI* – 6.17 FDI^2.

The term –44.65*FDI* signifies a linear decline in FDI over time. The negative coefficient, -44.65, implies that as FDI levels rise, this term leads to a proportional reduction in FDI. This decline might result from factors like diminishing returns on investment, rising costs, market saturation, or economic risks that deter further investment. Essentially, it reflects a natural tendency for FDI to decrease over time, assuming all other conditions remain unchanged.

The term –6.17 FDI^2 represents a quadratic decline in FDI, meaning that as FDI increases, the rate of decrease accelerates. The negative coefficient, –6.17, suggests that high levels of FDI can lead to even faster declines, potentially due to overinvestment or negative externalities like capital outflows or increased competition, which may outweigh the benefits of additional investment. This term underscores the risk of a self-reinforcing decline in FDI when it reaches excessively high levels.

The term +10.83*GDP* * *FDI* represents a positive interaction between GDP and FDI. The positive coefficient, +10.83, indicates that as GDP increases, it boosts FDI growth. This suggests that a robust and expanding economy attracts more investment, potentially due to factors like heightened investor confidence, improved infrastructure, or more favorable economic conditions that make the country a more appealing destination for foreign capital. This term acts as a counterbalance, suggesting that economic growth (represented by GDP) can offset the natural decline of FDI, potentially leading to an increase in FDI if GDP is sufficiently strong.

Ultimately, the equation $\frac{dFDI}{dt}$ –44.65 *FDI* – 6.17FDI^2 + 10.83 *GDP* * *FDI* models the dynamics of FDI, illustrating the factors that both decrease and increase investment over time. The linear and quadratic terms indicate a natural tendency for FDI to decline, especially at higher levels, possibly reflecting risks or diminishing returns. However, the positive interaction with GDP shows that economic growth can counteract this decline and even lead to increased FDI. The overall behaviour of FDI depends on the balance between these opposing forces, with GDP playing a vital role in sustaining or boosting foreign investment.

The Lotka-Volterra like model developed is a very special model. The model is very suitable to analyze dynamic interaction between two factors. Therefore, researcher can analyze the influence of FDI to GDP and GDP to FDI straight away using the same model at the same time.

RELIABILITY ANALYSIS OF THE PROPOSED LOTKA-VOLTERRA MODEL

We analyse the reliability of the propose Lotka-Volterra model. The model consists of two differential equations:

$$\frac{dGDP}{dt} = -12.65GDP + 3.36GDP^2 - 2.26GDP * FDI$$

$$\frac{dFDI}{dt} = -44.65FDI - 6.17FDI^2 + 10.83GDP * FDI$$

Variables and Parameters:

GDP: Gross Domestic Product

FDI: Foreign Direct Investment

The coefficients are –12.65, 3.36, –2.26, –44.65, –6.17, 10.83 are parameters that influence the rate of change of GDP and FDI.

Check for Stability:

The stability of the given system of ordinary differential equations (ODEs) describing the interaction between GDP and FDI is determined by analyzing the behavior of solutions near the system's equilibrium points. Specifically, the system is said to be stable at an equilibrium point (GDP∗, FDI∗) if small perturbations from this point result in trajectories that remain close to or return to (GDP∗, FDI∗) over time. Mathematically, this is often examined by evaluating the Jacobian matrix of the system at the equilibrium point and analyzing the eigenvalues: if all eigenvalues have negative real parts, the equilibrium is locally asymptotically stable; if any eigenvalue has a positive real part, the equilibrium is unstable.

Equilibrium Points: Set the right-hand side of both equations to zero and solve for GDP and FDI.

For ($\frac{dGDP}{dt}$ = 0); – 12.65*GDP* + 3.36GDP^2 – 2.26*GDP* * *FDI* = 0

Therefore, *GDP* (–12.65 + 3.36GDP – 2.26FDI) = 0

Thus, resulted in *GDP* = 0 or –12.65 + 3.36GDP – 2.26FDI = 0.

For (GDP = 0), *GDP* = 0 (trivial solution),

For the quadratic equation, –12.65 + 3.36GDP – 2.26FDI = 0 (nontrivial solution)

From the nontrivial solution, 3.36GDP = 12.65 + 2.26 FDI

GDP = $\frac{12.65 + 2.26 \text{ FDI}}{3.36}$

For ($\frac{dFDI}{dt}$ = 0); –44.65*FDI* – 6.17 FDI^2 + 10.83 *GDP* * *FDI* = 0

Therefore, *FDI* (–44.65 – 6.17 *FDI* + 10.83 *GDP)* = 0

For (FDI = 0), FDI = 0 (trivial solution).

or –44.65 – 6.17 *FDI* + 10.83 *GDP* = 0 (nontrivial solution)

From the nontrivial solution, –6.17*FDI* = 44.65 – 10.83

GDP
$FDI = \frac{44.65 - 10.83GDP}{-6.17}$
Considering the nontrivial solution,
$FDI = -\frac{44.65 - 10.83GDP}{6.17}$, but $GDP = \frac{12.65 + 2.26\,FDI}{3.36}$
Thus, $FDI = -\frac{44.65 - 10.83\left(\frac{12.65 + 2.26\,FDI}{3.36}\right)}{6.17} = 3.4782581$
Therefore, $GDP = \frac{12.65 + 2.26\,(3.4782581)}{3.36} = 6.1044236$

Linearize the System:

Near the equilibrium points, linearize the system using the Jacobian matrix. The Jacobian matrix is defined by,

$$J = \begin{bmatrix} \frac{\partial\left(\frac{dGDP}{dt}\right)}{\partial GDP} & \frac{\partial\left(\frac{dGDP}{dt}\right)}{\partial FDI} \\ \frac{\partial\left(\frac{dFDI}{dt}\right)}{\partial GDP} & \frac{\partial\left(\frac{dFDI}{dt}\right)}{\partial FDI} \end{bmatrix}$$

Substituting the partial derivatives gathered from the LV model into Jacobian matrix, we get:

$$J = \begin{bmatrix} -12.65 + 6.72GDP - 2.26FDI & -2.26GDP \\ 10.83FDI & -44.65 - 12.34FDI + 10.83GDP \end{bmatrix}$$

By substituting *GDP* = 6.1044236 and *FDI* = 3.4782581 (gathered in step 2) into Jacobian matrix above, resulted in

$$J = \begin{bmatrix} 20.510863 & -13.795997 \\ 37.669536 & -21.460798 \end{bmatrix}$$

Analyze Eigenvalues:

Using Scilab, we gather eigenvalues for the Jacobian matrix. The eigenvalues calculated are

-0.4749675 + 8.9041408i
-0.4749675 - 8.9041408i

Both real part of the eigenvalues gives negative values. The sign of the real part of the eigenvalues will indicate the stability of the equilibrium points; If all eigenvalues have negative real parts, the equilibrium is stable, otherwise the equilibrium is unstable (Hirsch, Smale & Devaney 2013). Since both real parts have negative values, hence, we conclude that the equilibrium is stable.

FOURTH ORDER RUNGE-KUTTA METHOD

The RK4 method is particularly valuable in scenarios where high precision is required, and the system being modeled exhibits complex, non-linear behaviour. The RK4 method used to relate GDP and FDI, as described by Hasan, Karim and Sulaiman (2015), is represented by (4) and (5). Let $y_1 = GDP$, $y_2 = FDI$, and $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$,

and let the function be

$$f_1(t, y_1, y_2) = -12.65y_1 + 3.36y_1^2 - 2.26y_1y_2$$

$$f_2(t, y_1, y_2) = -44.65y_1 - 6.17y_1^2 - 10.83y_1y_2$$

The RK4 update formula for given time step h, current time, t_i and current state $y_i = \begin{bmatrix} y_{1,i} \\ y_{2,i} \end{bmatrix}$ are

$$k_1 = f(t_i, y_i) = \begin{bmatrix} f_1(t_i, y_{1,i}, y_{2,i}) \\ f_2(t_i, y_{1,i}, y_{2,i}) \end{bmatrix}$$

$$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right)$$

$$k_4 = f(t_i + h, y_i + hk_3)$$

The update RK4 formula

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

In component form

$$GDP: y_{1,i+1} = y_{1,i} + \frac{h}{6}\left(k_{1,1} + 2k_{1,2} + 2k_{1,3} + k_{1,4}\right) \quad (4)$$

$$FDI: y_{2,i+1} = y_{1,i} + \frac{h}{6}\left(k_{2,1} + 2k_{2,2} + 2k_{2,3} + k_{2,4}\right) \quad (5)$$

We apply RK4 with $h = 0.00003$ for analyzing the *GDP* ↔ *FDI* interaction, represented by Equation (3). To compare the performance of the RK4 on Lotka-Volterra like model, we apply least-square models; with linear model (Mustafa & Idris 2021), Quadratic and Cubic model. The linear model coefficient in our model differ from linear model in Mustafa and Idris model (2021) may because of Mustafa and Idris (2021) apply on data from 1989 to 2018, while our linear least-square model use data from 2008 to 2020. Additionally, their model also include inflation and export. We construct all model compared in this paper via Scilab programming. The least-square models gathered from the dataset are summarize in Table 1.

RESULTS AND DISCUSSION

In the experiment, we applied RK4 method and various LS methods including linear, quadratic, and cubic to analyze the dynamic interaction between GDP and FDI using real-world data from The Malaysian Department of Statistic (DOSM 2023). The data spanned a period of 12 years, providing a robust foundation for modeling and comparison. The data is given in Table 2.

The RK4 method (Equations 4 and 5) was used to simulate the time evolution of GDP and FDI based on differential equations Lotka-Volterra like model derived from economic theory, while the LS methods were employed to fit polynomial models to the observed data, capturing the relationships between GDP and FDI in a static context. We develop Scilab programming for all models. The results are summarized in Tables 3-5 and Figures 1-4. Figures 1-3 compare observed and predicted graph for all least-square models. While Figure 4 plot the GDP and FDI predicted and observed value from Lotka-Volterra model. Table 3 compares GDP predicted values while Table 4 compares FDI predicted values from models developed. To analyze the accuracy for all models, five metrics for all models are given in Table 5.

We follow Pineiro et al. (2008) and Reddy and Henze (2023) technique to evaluate model accuracy beside RMSE and MAPE. The gradient coefficient, or slope, represents the relationship between observed and predicted values in a model. When the gradient coefficient is close to 1, as seen

TABLE 1. Least-square models gathered

LS model	GDP Independent, FDI dependent	FDI Independent, GDP dependent
Linear	$FDI = 1.4344006 + 0.158339GDP$	$GDP = 8.4473371 + 0.765659FDI$
Quadratic	$FDI = -14.43528 + 3.3323342GDP - 0.1501441GDP^2$	$GDP = 7.0647976 + 2.2044902FDI - 0.2823389FDI^2$
Cubic	$FDI = -17.747351 + 4.3888541GDP - 0.2581264GDP^2 + 0.0035455GDP^3$	$GDP = 0.4494166 + 15.105819FDI - 5.9146322FDI^2 + 0.6955542FDI^3$

TABLE 2. Malaysian GDP (in RM100 billion) and FDI (in RM10 billion) dataset from 2009-2020

Year	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
GDP	6.299	8.214	8.649	9.123	9.551	10.124	11.769	12.293	13.008	13.638	14.243	13.439
FDI	0.512	2.918	3.733	2.854	3.818	3.560	3.938	4.703	4.042	3.074	3.2364	1.4639

TABLE 3. Predicted GDP using Linear, Quadratic, Cubic Least-Square Models and RK4

Observed GDP values	Linear LS predicted GDP	Quadratic LS predicted GDP	Cubic LS predicted GDP	RK4 Predicted GDP
6.299	8.839355	8.119483	6.7264665	6.299
8.214	10.68153	11.09346	11.448451	8.214079
8.649	11.30554	11.35969	10.600361	8.649018
9.123	10.63253	11.05667	11.554238	9.123041
9.551	11.37062	11.36585	10.616495	9.551068
10.124	11.17308	11.33453	10.648475	10.12407
11.769	11.4625	11.36761	10.690357	11.76906
12.293	12.04823	11.18768	13.023907	12.29305
13.008	11.54213	11.36256	10.807753	13.00801
13.638	10.80097	11.17345	11.198772	13.63806
14.243	10.92532	11.24211	10.964922	14.243
13.439	9.568185	9.686898	12.069802	13.43906

in all models analyzed, this suggests a nearly one-to-one correspondence between observed and predicted values. Consequently, all models provide accurate predictions of GDP and FDI data.

The coefficient of determination r^2, quantifies how well the predicted values account for the variability in observed data. In the case of the GDP Lotka-Volterra model, an r^2 value of 1 indicates that the model perfectly explains the observed GDP variability, outperforming other models. For FDI, the Lotka-Volterra model shows a moderate relationship between predicted and observed values. However, the FDI Cubic and Quadratic models exhibit the highest r^2 values (0.63), suggesting they explain the variability in observed FDI better than the FDI Linear model. It is essential to recognize that a lower r^2 value does not necessarily imply that a model is ineffective; it may reflect a complex relationship between variables or the influence of unaccounted factors.

Root Mean Square Error (RMSE) measures the average magnitude of errors between predicted and observed values. The Lotka-Volterra models demonstrate the lowest RMSE (0.0065 for GDP and 0.7254 for FDI), indicating that they have the smallest average errors and are the most accurate among the models evaluated.

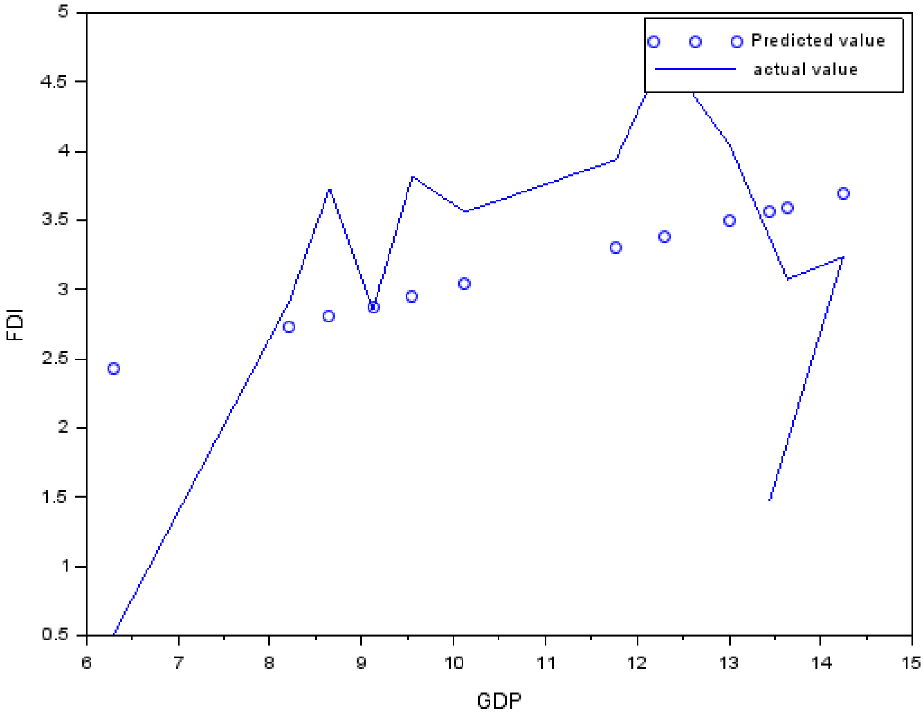
Mean Absolute Percentage Error (MAPE) assesses model accuracy as a relative percentage. The Lotka-Volterra models exhibit the lowest MAPE (0.00034 for GDP and 4.38494 for FDI), highlighting their superior accuracy in predicting both GDP and FDI.

TABLE 4. Predicted FDI using Linear, Quadratic, Cubic Least-Square Models and RK4

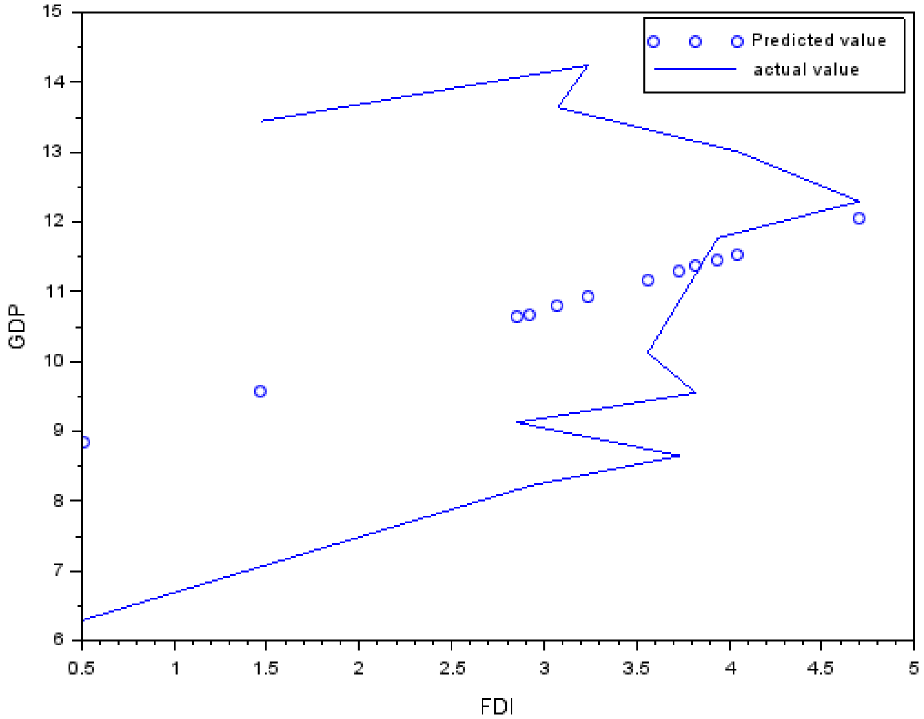
Observed FDI values	Linear LS predicted FDI	Quadratic LS predicted FDI	Cubic LS predicted FDI	RK4 Predicted FDI
0.512	2.431778	0.54238	0.5423796	0.512
2.918	2.734997	2.851874	2.8518743	3.038439
3.733	2.803875	3.196564	3.1965642	4.460037
2.854	2.878928	3.500637	3.5006366	2.936901
3.818	2.946697	3.712948	3.7129475	3.649384
3.560	3.037425	3.907696	3.9076958	3.921909
3.938	3.297893	3.931753	3.9317534	3.434363
4.703	3.380862	3.7838	3.7838001	3.615947
4.042	3.494075	3.469681	3.4696809	3.578161
3.074	3.593828	3.091165	3.0911653	3.563623
3.2364	3.689623	2.643125	2.6431245	3.526025
1.4639	3.562319	3.220678	3.2206778	3.48354

TABLE 5. Comparison of accuracy metrics for all models

Metrics	Linear Model		Quadratic Model		Cubic Model		RK4	
	GDP	FDI	GDP	FDI	GDP	FDI	GDP	FDI
Gradient (obs vs predicted)	1.0000	1	0.9999	1.0000	0.9999	1	0.9999	0.90622
Intersection (obs vs predicted)	-5.0E-04	-1.3E-06	0.000699	-2.4E-07	0.000198	-6.1E-09	9.8E-04	0.1547
r^2 (obs vs predicted)	0.1212	0.1212	0.1483	0.6300	0.3423	0.6300	1	0.567
RMSE	2.2846	1.0389	2.2488	0.6928	1.3144	0.6742	0.0065	0.7254
MAPE	19.651	56.389	19.964	20.441	14.396	3.8870	0.00034	4.38494

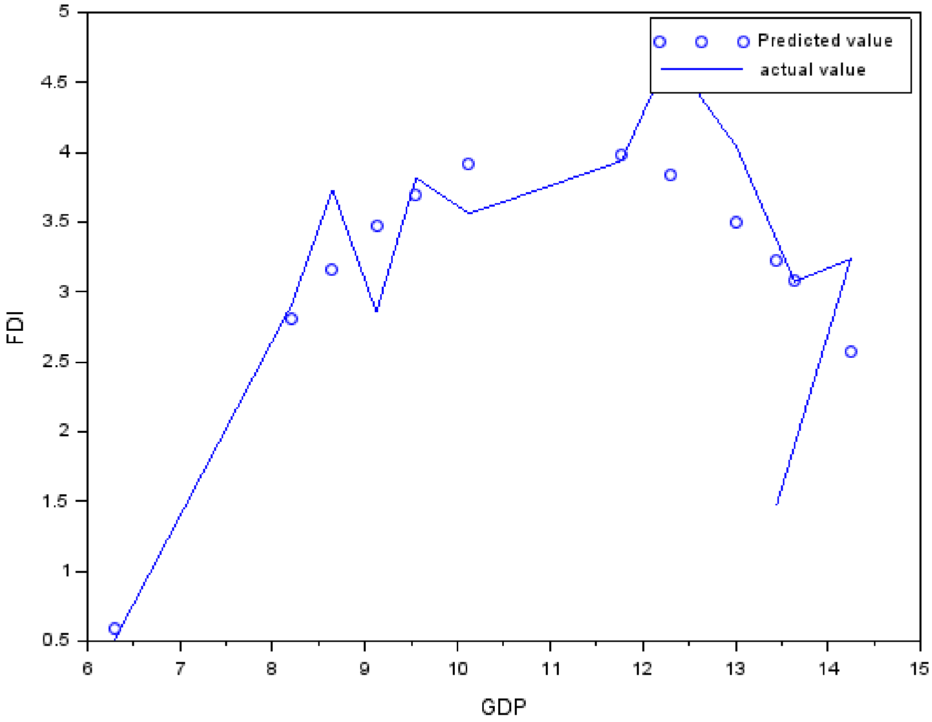


(a)



(b)

FIGURE 1. Comparison of observed and predicted values for all Least-Square models: (a) FDI vs GDP for FDI Linear Model (b) GDP vs FDI for GDP Linear Model



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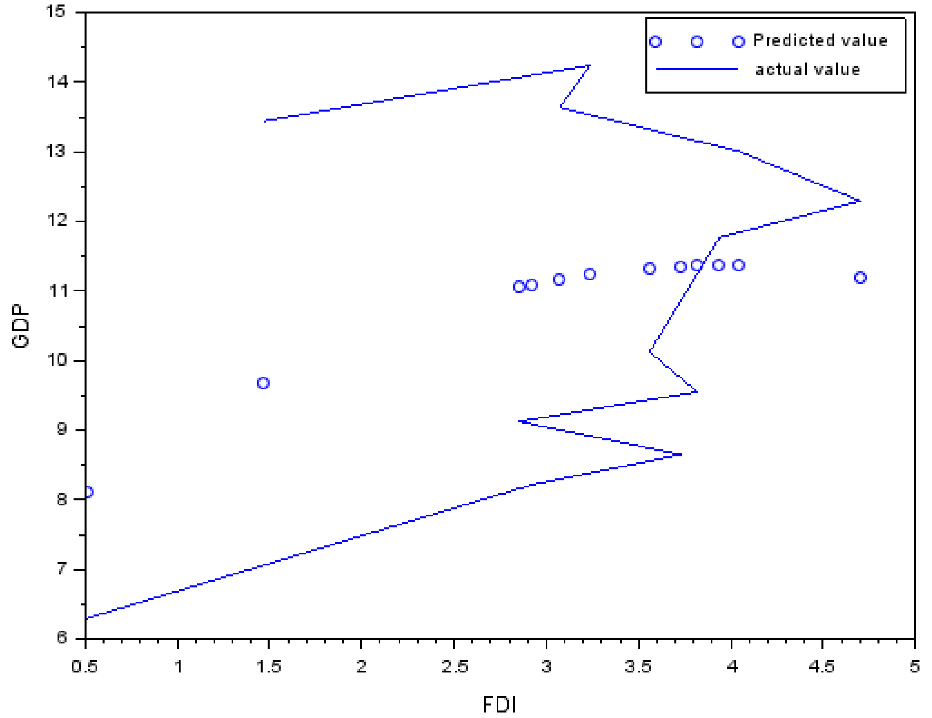

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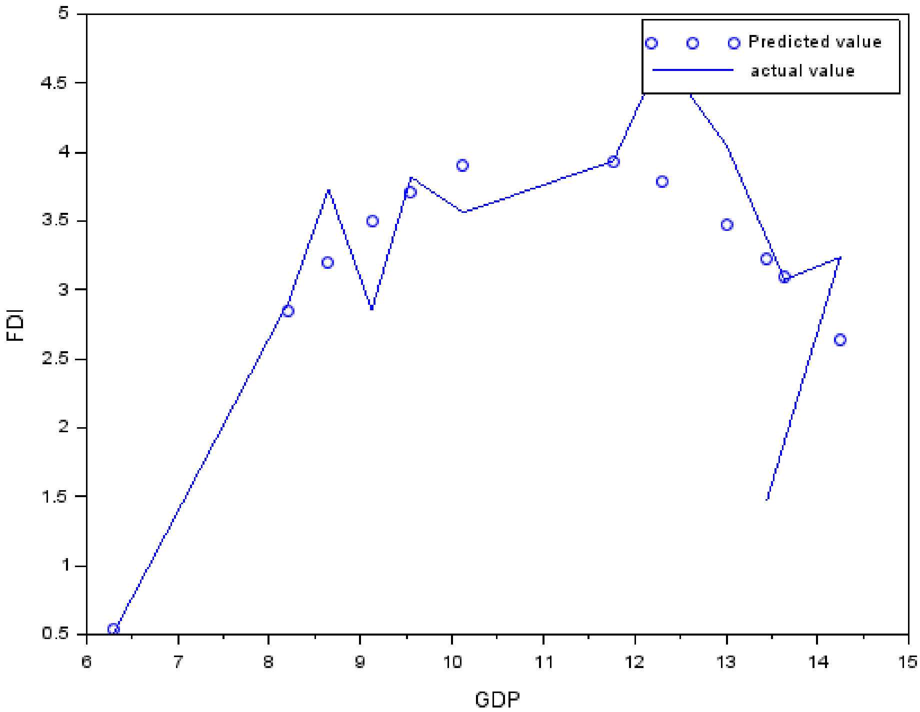
FIGURE 2. Comparison of observed and predicted values for all Least-Square models: (a) FDI vs GDP for FDI Quadratic Model (b) GDP vs FDI for GDP Quadratic Model



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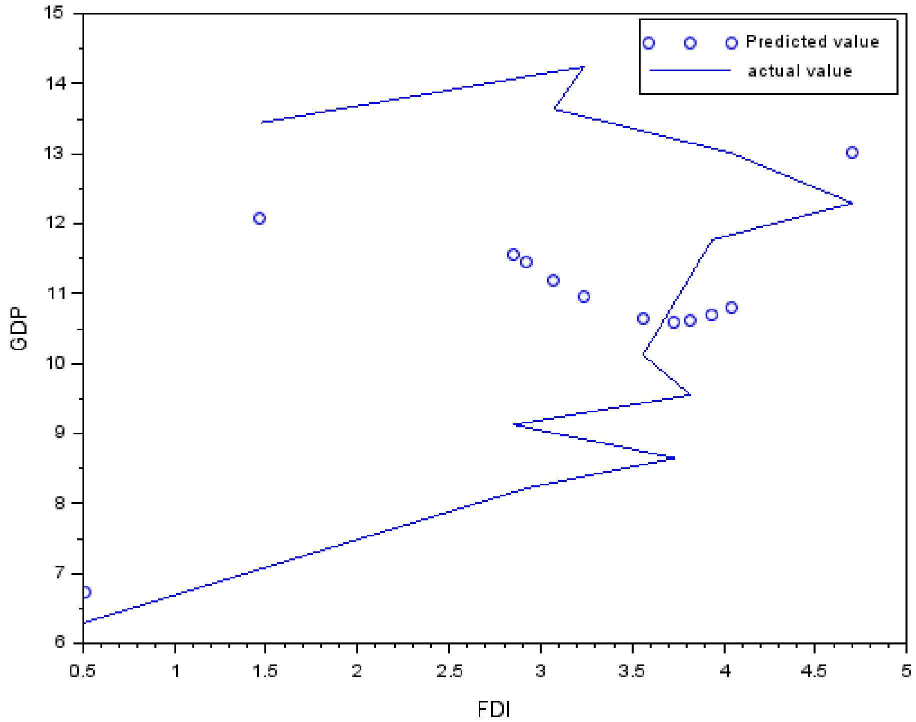

(b)

FIGURE 3. Comparison of observed and predicted values for all Least-Square models: (a) FDI vs GDP for FDI Cubic Model (b) GDP vs FDI for GDP Cubic Model

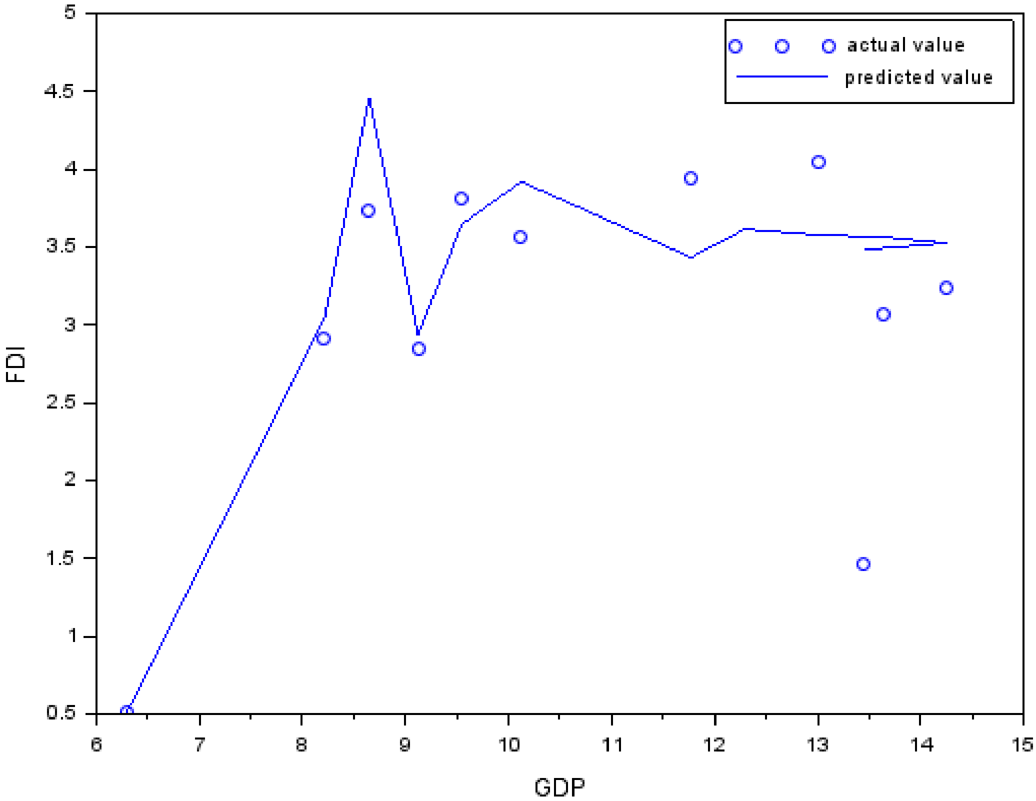

FIGURE 4. Predicted and observed FDI vs GDP for RK4 method

Overall, based on these metrics, the Lotka-Volterra model outperforms the least-square models in explaining variability (r^2), accuracy (RMSE), and predictive accuracy (MAPE). This implies that the dynamic interactions captured by the Lotka-Volterra model best represent the relationship between GDP and FDI, supporting the initial analysis of the underlying equations.

The results showed distinct advantages for each method. The RK4 method excelled in accurately modeling the dynamic interaction over time, particularly capturing the non-linear and complex feedback mechanisms between GDP and FDI. In contrast, the least-square methods provided useful insights into the general trends and patterns within the data. Overall, the RK4 method demonstrated superior accuracy in dynamic modeling, while the least-square methods, particularly the quadratic and cubic models, balanced fit quality with generalizability, making them effective for static analysis of the GDP-FDI relationships.

CONCLUSIONS

The analysis indicates that the Lotka-Volterra model offers superior performance compared to traditional least-square models in predicting GDP and FDI data (Tables 3-5, Figures 1-4). Its high r^2 values for both GDP and FDI observed vs predicted relation with almost 1:1 mapping of observed vs predicted values, low RMSE, and low MAPE demonstrate its accuracy and ability to capture the complex dynamics between GDP and FDI. These metrics suggest that the Lotka-Volterra model effectively explains the variability in observed values and provides accurate predictions, particularly for GDP. While the FDI Cubic and Quadratic models show relatively strong r^2, values, the overall accuracy and predictive capabilities of the Lotka-Volterra model make it the most reliable choice for both GDP and FDI predictions.

Despite the strong performance of the Lotka-Volterra model, it is important to acknowledge that a lower r^2, value does not inherently diminish a model's usefulness. Instead, it may highlight the complexity of relationships between variables or the presence of additional influencing factors not accounted for in the model. Therefore, while the Lotka-Volterra model is the most effective based on the current analysis, further exploration and refinement of other models could enhance their ability to capture the intricacies of GDP and FDI interactions. In conclusion, the dynamic interaction approach of the Lotka-Volterra model aligns well with the underlying economic relationships, making it a robust tool for understanding and predicting GDP and FDI trends.

The Lotka-Volterra-like model offers valuable insights into the non-linear, dynamic nature of GDP and

FDI interactions. This approach allows policymakers and economists to better understand how these two variables influence each other over time and under different conditions. By simulating the system, one can explore potential outcomes and identify strategies to optimize the positive impacts of FDI on economic growth while mitigating any adverse effects.

In conclusion, using a Lotka-Volterra-like model combined with the RK4 method provides a robust framework for analysing the dynamic interaction between GDP and FDI. It offers a powerful tool for exploring complex economic relationships and making informed decisions to promote sustainable economic development.

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