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Statistical Methods by a Reparameterization of the Coefficient of Variation for a Three-Parameter Lognormal Model: An Application to Thailand Rainfall Kinetic Energy Data

(Kaedah Statistik melalui Penyusunan Semula Pekali Variasi untuk Model Lognormal Tiga Parameter: Aplikasi untuk Data Tenaga Kinetik Hujan Thailand)

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ABSTRACT

In the study, a crucial research gap concerning the estimation of the coefficient of variation in an asymmetric distribution was addressed, specifically focusing on the three-parameter lognormal (3PLN) model, which exhibits large variation. The purpose of this research was to determine the effectiveness of four statistical approaches: likelihood-based, parametric bootstrap, profile likelihood, and Bayesian inference in formulating confidence intervals for reparameterizing the coefficient of variation (CV) within the 3PLN model. To evaluate the performance of the confidence intervals, we utilize specific performance measures such as the coverage rate and mean length. The results of our simulation study provide insights into how the profile likelihood method performs in estimating the CV, showing its effectiveness compared to alternative methods, particularly in ensuring accurate coverage rates. Nevertheless, when utilizing Adam's optimization algorithm to derive point estimates, the Bayesian approach stands out as a dependable option for establishing confidence intervals, especially when dealing with a wide range of variances, from small to large. To demonstrate the practical application of our research, we utilize all proposed confidence intervals in estimating rainfall kinetic energy data. This real-world application serves as validation and showcases the utility of confidence intervals in practical scenarios.

Keywords: Adam optimization algorithm; Bayesian inference; parametric bootstrap; profile likelihood; rainfall kinetic energy

ABSTRAK

Dalam penyelidikan ini, jurang kajian penting mengenai anggaran pekali variasi dalam taburan asimetri, khususnya model lognormal tiga parameter (3PLN) yang menunjukkan variasi yang besar, ditekankan. Tujuan penyelidikan ini adalah untuk menentukan keberkesanan empat pendekatan statistik: berasaskan kebolehjadian, butstrap berparamater, kebolehjadian profil dan inferens Bayesian dalam merumuskan selang keyakinan untuk penyusunan semula pekali variasi (CV) dalam model 3PLN. Untuk menilai prestasi selang keyakinan, kami menggunakan ukuran prestasi khusus seperti kadar liputan dan panjang min. Keputusan simulasi memberikan pandangan tentang bagaimana prestasi kaedah kebolehjadian profil dalam menganggar CV, menunjukkan keberkesanannya berbanding kaedah alternatif, terutamanya dalam memastikan kadar liputan yang tepat. Namun begitu, apabila menggunakan algoritma pengoptimuman Adam untuk memperoleh anggaran titik, pendekatan Bayesian menonjol sebagai pilihan yang boleh dipercayai untuk mewujudkan selang keyakinan, terutamanya apabila berurusan dengan pelbagai variasi, dari kecil hingga besar. Untuk menunjukkan aplikasi praktikal penyelidikan ini, semua selang keyakinan yang dicadangkan dalam menganggar data tenaga kinetik hujan digunakan. Aplikasi dunia sebenar ini berfungsi sebagai pengesahan dan menunjukkan kegunaan selang keyakinan dalam senario praktikal.

Kata kunci: Algoritma pengoptimuman Adam; butstrap berparameter; inferens Bayesian; kebolehjadian profil; tenaga kinetik hujan

INTRODUCTION

Soil quality is negatively impacted by strong winds, heavy rain, and flowing water. These factors contribute to soil erosion, which is the relationship between observed rainfall intensity and the resulting kinetic energy during rainfall events. Kinetic energy is influenced by a spectrum of raindrop sizes, which in turn depends on the nature of the distribution of these sizes (Fox 2004). Raindrop sizes are typically measured by their diameter. Notably, rainfall kinetic energy serves as an indicator to assess the rain's potential to detach soil, and it plays a fundamental role in soil erosion studies (Mineo et al. 2019). Understanding the variation in rainfall kinetic energy through the coefficient of variation is crucial for making informed decisions regarding soil erosion management. When this variation is large, it indicates the presence of more intense rainfall events that can cause severe soil detachment and increase the risk of crop damage. In our current study, daily raindrop sizes are recorded and collected by Land Development Regional Office 6 located in Chiang Mai, Northern Thailand. After analyzing the distribution fitting, we found that daily rainfall kinetic energy follows the properties of a three-parameter lognormal (3PLN) model.

A 3PLN distribution is an asymmetric model that accurately represents highly right-skewed data, which cannot be effectively modeled by a two-parameter lognormal distribution, also known as a lognormal distribution (Singh, Cruise & Ma 1990). In the 3PLN model, a threshold parameter is defined as a lower bound for the data, added to the mean (μ) and variance (σ^2) parameters of the lognormal model. When the threshold is set to zero, the 3PLN model reduces to a lognormal model. Furthermore, to obtain a symmetric distribution, the random variable of the lognormal model is log-transformed to create a normal model. The 3PLN distribution is found to have extensive application in various research areas involving real-life data. In operational hydrology, Charbeneau (1978) employed this model for in-stream flow synthesis. Chen and Miao (2012) and Cohen and Whitten (1980) utilized it to manipulate maximum flood levels in the Susquehanna River at Harrisburg, Pennsylvania. In industrial factories, Chen (2006), and Chen and Miao (2012) employed the 3PLN model to analyze plastic laminate strengths in the manufacturing process. Additionally, the model is fitted and utilized in fatigue life data analysis, as demonstrated in Cohen and Whitten (1980), Cohen, Whitten and Ding (1985), and McCool (1975).

The coefficient of variation (CV) is a statistical measure used to assess the relative dispersion of data points around the mean in a data series. It provides a way to express the variability of the data in relation to the mean. The CV can also be reparameterized to represent a function of specific indicators of interest, which facilitates the interpretation of the probability density function. In the context of evaluating inequality among populations, Bendel et al. (1989) suggest that the CV is often preferred over the Gini coefficient as a measure of relative precision. This is because the CV is more sensitive to individuals in a positively skewed distribution. Interval estimation, also known as confidence interval (CI) estimation, is a statistical method used to estimate a parameter of interest θ in probability and statistical inference. It provides a range of values, with lower and upper limits, that cover the true value of the parameter θ at a specified confidence level of $100(1-\alpha)\%$. Here, α represents the probability of rejecting the true null hypothesis (Type I error) that is acceptable. Casella and Berger (2002) provide evidence supporting the use of CIs in statistical inference.

The likelihood function of a 3PLN model incorporates various features when including the point estimates of its parameters. Wingo (1984) devised a numerical algorithm that maximizes the conditional log-likelihood function, suitably transformed, to obtain the local maximum likelihood (LML) estimate. Hirose (1997) developed an algorithm for LML estimation that combines the continuation method and the extended lognormal model, without requiring careful selection of initial values. Komori and Hirose (2004) proposed a reparameterization approach to obtain parameter estimates for a 3PLN model. They designed an algorithm based on the bisection method to search for the profile likelihood function, which is then maximized. Vera and Díaz-García (2008) solved the problem of finding the threshold value that maximizes the log-likelihood function and equals zero by using a global simulated annealing optimization heuristic. In a different approach, Nagatsuka and Balakrishnan (2012) employed invariant statistics with unknown locations to estimate the 3PLN parameters efficiently, without considering the unbounded likelihood function. Subsequently, Komori (2015) developed an algorithm associated with a parameterization similar to Vera and Díaz-García (2008), which exhibited improved performance.

In the past, several studies have focused on constructing CIs for parameters in 3PLN distribution. For instance, Griffiths (1980) presented a CI based on the likelihood function, utilizing tangents to the contour that encloses an approximate 95% confidence region. Chen and Miao (2012) subsequently developed exact CIs and exact upper confidence limits for the threshold parameter. More recently, Maneerat, Niwitpong and Nakjai (2022) proposed the use of a generalized pivotal quantity to estimate the median of a three-parameter lognormal distribution. However, their CIs are limited in terms of accommodating large variations and small sample sizes.

As pointed out by academic researchers, we address a significant research gap regarding the large variance in a 3PLN model. The objective of this paper was to present methods for formulating CIs for a reparameterization of the CV in a 3PLN model. These methods include the likelihood-based, parametric bootstrap, profile likelihood, and Bayesian approaches, detailed in Statistical Methods section. To determine the most effective CI, we evaluate the performance of all proposed intervals using performance measures such as the coverage rate and mean length. These evaluations are presented in the Confidence Interval Performance section. Notably, in real-world applications, we apply the proposed CIs to rainfall kinetic energy as a demonstration, as described in Real Example section. Finally, this paper concludes with a brief discussion and summary in the Discussion and Conclusion sections, respectively.

STATISTICAL METHODS

Suppose that independent and identically distributed (iid) sample $X = (X_1, \ldots, X_n)$ is drawn from a 3PLN distribution which is governed by parameters (β, μ_X, ϑ_X^2), where the parameter β is a lower bound of X. The random variable X is lognormally distributed $[W = X - \beta \sim LN(\mu_W, \vartheta_W^2)]$ if the Y = ln$W$ is normally distributed, denoted as Y ~ N (ω, τ), where $\omega = \exp(\mu_Y)$ and $\tau = \exp(\vartheta_Y)$. The probability density function of Y is defined as

$$g(x;\beta,\omega,\vartheta_y) = \left[(x-\beta)\sqrt{2\pi \ln^2\tau}\right]^{-1} \exp\left\{-(2\ln^2\tau)^{-1}\left[\ln\left(\frac{x-\beta}{\omega}\right)\right]^2\right\}$$

for $x > \beta$, $0 \le \beta < \infty$, and (ω, τ) > 0. The CV of X is $\delta = \vartheta_X/\mu_X$, where the standard deviation is $\vartheta_X = \omega\{\exp(\ln^2\tau)[\exp(\ln^2\tau) - 1]\}^{1/2}$ and the mean is $\mu_X = \beta + \omega\exp(\ln^2\tau/2)$. Then, the CV is log-transformed, expressed as

$$\ln(\delta) = \ln\omega + \frac{1}{2}\left[2\ln^2\tau + \ln\left(1 - \frac{1}{\exp(\ln^2\tau)}\right)\right] - \ln(\omega\exp[(\ln^2\tau)/2]) - \ln\left(1 + \frac{\beta}{\omega\exp[(\ln^2\tau)/2]}\right) \quad (1)$$

As τ approaches infinity, the term $\lim_{\tau\to\infty}\ln\left(1 - \frac{1}{\exp(\ln^2\tau)}\right) = 0$, and then the log-transformation of CV (1) is reparametrized, written as

$$\kappa = \frac{1}{2}\ln^2\tau - \ln\left(1 + \frac{\beta}{\omega\exp[(\ln^2\tau)/2]}\right) \quad (2)$$

which is our parameter of interest for estimating in the current study. After that, it is approximated by statistical interval methods, which are detailed in the following section. Considering that X is an iid random samples of size n from 3PLN (β, ω, lnτ), and the log-likelihood function for $\theta = (\beta, \omega, \ln\tau)$ is

$$l(\theta) = -\frac{n}{2}[\ln(2\pi) + \ln(\ln^2\tau)] - \sum_{i=1}^n \ln(x_i - \beta)^{-1} - \frac{1}{2\ln^2\tau}\sum_{i=1}^n \ln^2\left(\frac{x_i - \beta}{\omega}\right) \quad (3)$$

The maximum likelihood estimates (MLEs) for $\theta = (\beta, \omega, \ln\tau)$ can be deserved by solving the first derivatives of the log-likelihood (3) and setting them to zero as follows:

$$\begin{aligned}
\frac{\partial l(\theta)}{\partial\beta} &= (\ln^2\tau)^{-1}\sum_{i=1}^n [(x_i-\beta)^{-1}\ln(x_i-\beta)] - [(\ln^2\tau)^{-1}\ln\omega - 1]\sum_{i=1}^n (x_i-\beta)^{-1} = 0 \\
\frac{\partial l(\theta)}{\partial\omega} &= (\omega\ln^2\tau)^{-1}\left[\sum_{i=1}^n \ln(x_i-\beta) - n\ln\omega\right] = 0 \\
\frac{\partial l(\theta)}{\partial\ln\tau} &= (-\ln\tau)^{-1}\left\{n - \frac{1}{\ln^2\tau}\sum_{i=1}^n \ln^2\left(\frac{x-\beta}{\omega}\right)\right\} = 0
\end{aligned} \quad (4)$$

From Equation (4), we obtain the MLE of lnω as

$$\ln\hat{\omega} = n^{-1}\sum_{i=1}^n \ln(x_i - \beta) \quad (5)$$

and then the MLE of $\ln^2\tau$ becomes

$$(\widehat{\ln\tau})^2 = n^{-1}\sum_{i=1}^n [\ln(x_i-\beta)]^2 - n^{-2}\left(\sum_{i=1}^n \ln(x_i - \hat{\beta})\right)^2 \quad (6)$$

Replacing the Equations (5)-(6) in the log-likelihood function (3), the profile likelihood of β is

$$P(\beta) = -\frac{n}{2}\ln\{2\pi(n^{-1}\sum_{i=1}^n [\ln(x_i-\beta)]^2 - n^{-2}[\sum_{i=1}^n \ln(x_i-\beta)]^2)\} - \sum_{i=1}^n \ln(x-\beta)^{-1} - \frac{\sum_{i=1}^n\left[\ln\left(\frac{x-\beta}{n^{-1}\sum_{i=1}^n \ln(x_i-\beta)}\right)\right]^2}{2\left\{n^{-1}\sum_{i=1}^n[\ln(x_i-\beta)]^2 - n^{-2}\left[\sum_{i=1}^n \ln(x_i-\beta)\right]^2\right\}} \quad (7)$$

which is maximized for obtaining the MLE of β, denoted as $\hat{\beta}$. It can be seen that Equation (7) cannot solve to find the explicit form of β, the Adam optimization algorithm is raised to obtain its MLE.

LIKELIHOOD-BASED METHOD

This method involves formulating an interval by using the likelihood function to solve for the Fisher information matrix, thereby obtaining the variance. Subsequently, a likelihood-based interval for κ is constructed using a normal approximation. For the log-likelihood given in Equation (3), the Fisher information matrix for θ is expressed as follows:

$$I(\theta) = -E\begin{bmatrix} \frac{\partial l^2(\theta)}{\partial\beta^2} & \frac{\partial l^2(\theta)}{\partial\beta\partial\omega} & \frac{\partial l^2(\theta)}{\partial\beta\partial\ln\tau} \\ \frac{\partial l^2(\theta)}{\partial\omega\partial\beta} & \frac{\partial l^2(\theta)}{\partial\omega^2} & \frac{\partial l^2(\theta)}{\partial\omega\partial\ln\tau} \\ \frac{\partial l^2(\theta)}{\partial\ln\tau\partial\beta} & \frac{\partial l^2(\theta)}{\partial\ln\tau\partial\omega} & \frac{\partial l^2(\theta)}{\partial\ln^2\tau} \end{bmatrix} = \begin{bmatrix} \frac{(1+\ln^2\tau)\exp(\ln^2\tau)}{\omega^2\ln^2\tau} & \frac{\sqrt{\exp(\ln^2\tau)}}{\omega^2\ln^2\tau} & \frac{-2\sqrt{\exp(\ln^2\tau)}}{\omega^2\ln\tau} \\ \frac{\sqrt{\exp(\ln^2\tau)}}{\omega^2\ln^2\tau} & \frac{1}{\omega^2\ln^2\tau} & 0 \\ \frac{-2\sqrt{\exp(\ln^2\tau)}}{\omega^2\ln\tau} & 0 & \frac{2}{\ln^2\tau} \end{bmatrix}$$

which is motivated by Cohen (1951), where the second derivatives of θ are determined as follows:

$$\frac{\partial l^2(\theta)}{\partial\beta^2}=\frac{1}{\ln\tau}\sum_{i=1}^{n}\frac{\ln(x_i-\beta)-1}{(x_i-\beta)^2}-\left(\frac{\ln\omega}{\ln\tau}-1\right)\sum_{i=1}^{n}(x_i-\beta)^{-2}$$

$$\frac{\partial l^2(\theta)}{\partial\omega^2}=\frac{1}{\omega^2\ln^2\tau}\left[-\frac{n}{\ln\tau}-\ln\tau\left(\sum_{i=1}^{n}\ln(x_i-\beta)-n\ln\omega\right)\right]$$

$$\frac{\partial l^2(\theta)}{\partial\ln\tau}=\frac{n}{\ln^2\tau}-\frac{3}{\ln^4\tau}\sum_{i=1}^{n}\ln^2\left(\frac{x-\beta}{\omega}\right)$$

$$\frac{\partial l^2(\theta)}{\partial\beta\partial\omega}=-(\omega\ln^2\tau)^{-1}\sum_{i=1}^{n}(x_i-\beta)^{-1}$$

$$\frac{\partial l^2(\theta)}{\partial\beta\partial\ln\tau}=-2(\ln^3\tau)^{-1}\sum_{i=1}^{n}[(x_i-\beta)^{-1}\ln(x_i-\beta)]+[2(\ln^3\tau)^{-1}\ln\omega]\sum_{i=1}^{n}(x_i-\beta)^{-1}$$

$$\frac{\partial l^2(\theta)}{\partial\omega\partial\ln\tau}=-2(\omega\ln^3\tau)^{-1}\left[\sum_{i=1}^{n}\ln(x_i-\beta)-n\ln\omega\right]$$

Therefore, it can be constructed the 100 $(1-a)$ % likelihood-based confidence interval for κ, expressed as

$$l_L=\hat{\kappa}+Z_{\alpha/2}\sqrt{V_{\hat{\kappa}}}$$

$$u_L=\hat{\kappa}+Z_{1-\alpha/2}\sqrt{V_{\hat{\kappa}}}$$

where $P(Z_{a/2}\le Z\le Z_{1-a/2})=1-a$ and Z_a denotes the a^{th} quantile of standard normal distribution: $N(0,1)$. It can be seen that the asymptotic distribution of $\hat{\theta}$ is a multivariate normal with a mean vector of θ and a variance-covariance matrix V; specially it is

$$V=\frac{1}{nI(\theta)}=\begin{bmatrix}\frac{\omega^2\ln^2\tau}{n\mathrm{B}\exp(\ln^2\tau)} & \frac{\omega\ln^2\tau}{n\mathrm{B}\sqrt{\exp(\ln^2\tau)}} & \frac{\omega\ln^3\tau}{n\mathrm{B}\sqrt{\exp(\ln^2\tau)}}\\ \frac{\omega\ln^2\tau}{n\mathrm{B}\sqrt{\exp(\ln^2\tau)}} & \frac{\omega^2\ln^2\tau}{n}\left[1+\frac{1}{\mathrm{B}}\right] & \frac{\omega\ln^3\tau}{n\mathrm{B}}\\ \frac{\omega\ln^3\tau}{n\mathrm{B}\sqrt{\exp(\ln^2\tau)}} & \frac{\omega\ln^3\tau}{n\mathrm{B}} & \frac{\ln^2\tau}{n}\left[1+\frac{2\ln^2\tau}{\mathrm{B}}\right]\end{bmatrix}$$

where $\mathrm{B}=\exp(\ln^2\tau)(1+\ln^2\tau)-2\ln^2\tau-1$. Since κ is the function of θ, $\hat{\kappa}$ follows an asymptotic normal distribution with mean κ and variance $V_{\hat{\kappa}}$, denoted as $\hat{\kappa}\sim\mathrm{N}(\kappa,V_{\hat{\kappa}})$;

$$V_{\hat{\kappa}}=\left(\frac{\partial\kappa}{\partial\theta}\right)V\left(\frac{\partial\kappa}{\partial\theta}\right)\tag{8}$$

where $\left(\frac{\partial\kappa}{\partial\theta}\right)=\left(\frac{\partial\kappa}{\partial\beta};\frac{\partial\kappa}{\partial\omega};\frac{\partial\kappa}{\partial\ln\tau}\right)=\left(-\frac{1}{\mu_X};\ \frac{\beta}{\omega\mu_X};\ \ln\tau\left[1+\frac{\beta}{\mu_X}\right]\right)$. Equation (8) is derived based on the variance behavior theorem as described in Casella and Berger (2002).

PARAMETRIC BOOTSTRAP METHOD

Efron (1979) first introduced a statistical technique; that is, the empirical bootstrap. Conceptually, the bootstrap idea is to formulate computation on the data itself to estimate the variation of statistics computed from the same data. The parametric bootstrap is also a statistical technique that generates the bootstrap samples from the underlying distribution (a parametrized distribution). The sources of bootstrap samples make the difference between the empirical and parametric bootstraps. Modern computing power can make the bootstrap that would be feasible to perform computations effectively. Our idea of the Bootstrap will be to estimate the variation of the point estimate for the reparameterized CV (2). There is a CI-based bootstrap method for estimating κ as follows:

The Acceleration bias-corrected percentile interval (ABCP) interval is formulated from the percentile of bootstrap distribution. It depends on the acceleration factor, expressed as

$$\widehat{af}=\sum_{i=1}^{n}[\hat{\kappa}_{(.)}-\hat{\kappa}_{(i)}]^3\left\{\left(\sum_{i=1}^{n}[\hat{\kappa}_{(.)}-\hat{\kappa}_{(i)}]^2\right)^{-3/2}/6\right\}$$

For each i in the range of $[n]=1,2...,n$, the mean $\hat{\kappa}_{(.)}$ is calculated for the Jackknife sub-sample consisting of all data points, except the i^{th} one, expressed as

$$\hat{\kappa}_{(i)}=(n-1)^{-1}\sum_{j\in[n],j\ne i}x_j$$

Note that $\hat{\kappa}_{(i)}$ be an estimate of κ based on Jackknife replications which are systematically leaving out each observation from a dataset as follows.

$$\hat{\kappa}_{(1)}\equiv s(x_2,x_3,\dots,x_n)$$

$$\hat{\kappa}_{(2)}\equiv s(x_1,x_3,\dots,x_n)$$

$$\hat{\kappa}_{(n)}\equiv s(x_2,x_3,\dots,x_{n-1})$$

which can be written as $\hat{\kappa}_{(i)}\equiv s(x_1,x_2,\dots,x_{i-1},x_{i+1},\dots,x_n)$. Then, the average of these n Jackknife replicates

$$\hat{\kappa}_{(.)}=n^{-1}\sum_{i=1}^{n}\hat{\kappa}_{(i)}$$

Therefore, the 100 $(1-a)$ % ABCP interval for κ is formed by $\left[\hat{\kappa}^*_{\phi(\alpha_1^{(A)})},\hat{\kappa}^*_{\phi(\alpha_2^{(A)})}\right]$ where $\phi(.)$ be the commutative density function of standard normal, denoted as $N(0,1)$, and then

$$\alpha_1^{(A)}=b_0+\frac{b_0+Z_{\alpha/2}}{1-\widehat{af}(b_0+Z_{\alpha/2})}$$

$$\alpha_2^{(A)}=b_0+\frac{b_0+Z_{1-\alpha/2}}{1-\widehat{af}(b_0+Z_{1-\alpha/2})}$$

where $b_0 = \phi^{-1}(P_0)$ be the bias correction factor; $P_0 = \#\{\hat{\kappa}_r^* \leq \hat{\kappa}_{mle}\}/B$; $r = 1,2,..., B$ and Z_a be the $100(a)^{th}$ percentile of $N(0,1)$

PROFILE LIKELIHOOD METHOD

The interval is constructed based on the asymptotic chi-square distribution of the log-likelihood ratio test as the profile likelihood-based interval. Importantly, the PL can be served better when the likelihood function is asymmetric about the MLE such that it is considered in this study.

Recall that the CV (2) is approximately $\kappa = \omega\exp(\ln^2\tau)/[\beta + \omega\sqrt{\exp(\ln^2\tau)}]$. Given the vector of observations $x = (x_1, x_2,..., x_n)'$, we obtain $\hat{\kappa}$ by substituting $(\hat{\beta}, \hat{\omega}, \widehat{\ln\tau})$, denoted as $(\beta(\kappa), \omega(\kappa), \ln\tau(\kappa))$. The profile-likelihood for κ is

$$\mathrm{R}(\kappa) = \ln\left[\frac{\max_{\kappa} L(\kappa,\beta(\kappa),\omega(\kappa),\ln\tau(\kappa)|x)}{L(\hat{\kappa},\hat{\omega},\widehat{\ln\tau}|x)}\right] = l(\kappa,\beta(\kappa),\omega(\kappa),\ln\tau(\kappa)|x) - l(\hat{\kappa},\hat{\omega},\widehat{\ln\tau}|x)$$

where $l(\kappa, \beta(\kappa), \omega(\kappa), \ln\tau(\kappa)|x)$ is the profile log-likelihood for κ deriving from the following equation is

$$l(\kappa|x) = c' - \frac{n}{2}\ln(\ln^2[\tau(\kappa)]) + \sum_{i=1}^{n}\ln\left[x_i - \frac{\omega(\kappa)\sqrt{\ln^2[\tau(\kappa)]}\left(\sqrt{\ln^2[\tau(\kappa)]}-\kappa\right)}{\kappa}\right]$$
$$-(2\ln^2[\tau(\kappa)])^{-1}\textstyle\sum_{i=1}^{n}\left\{\ln\left[x_i - \frac{\omega(\kappa)\sqrt{\ln^2[\tau(\kappa)]}\left(\sqrt{\ln^2[\tau(\kappa)]}-\kappa\right)}{\kappa}\right] - \ln\omega(\kappa)\right\}^2$$

From the Equations (5)-(6), the MLEs of $(\omega, \ln\tau)$ can be expressed in term of $\hat{\beta}$ as follows:

$$\ln\hat{\omega}_{mle} = n^{-1}\textstyle\sum_{i=1}^{n}\ln(x_i - \hat{\beta})$$

$$(\widehat{\ln\tau}_{mle})^2 = (n-1)n^{-1}\left\{(n-1)^{-1}\sum_{i=1}^{n}\left[\ln(x_i-\hat{\beta}) - \left(n^{-1}\sum_{i=1}^{n}\ln(x_i-\hat{\beta})\right)\right]^2\right\}$$

Then, the profile log-likelihood function is based on MLEs $(\hat{\kappa}, \hat{\omega}_{mle}, \hat{\tau}_{mle})$, written as

$$l(\hat{\kappa}|x) = c'' - \frac{n}{2}\ln\left\{\frac{(n-1)}{n}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left[\ln(x_i-\hat{\beta}) - \left(\frac{1}{n}\sum_{i=1}^{n}\ln(x_i-\hat{\beta})\right)\right]^2\right]\right\}$$
$$+\sum_{i=1}^{n}\ln(x_i-\hat{\beta}) - \left\{\left(\frac{2(n-1)}{n}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left[\ln(x_i-\hat{\beta}) - \left(\frac{1}{n}\sum_{i=1}^{n}\ln(x_i-\hat{\beta})\right)\right]^2\right]\right)^{-1}\right.$$
$$\left.\times\sum_{i=1}^{n}\left[\ln(x_i-\hat{\beta}) - \left(\frac{1}{n}\sum_{i=1}^{n}\ln(x_i-\hat{\beta})\right)\right]^2\right\}$$

where c' and c'' are constants. Define the function

$$\gamma(\kappa) = 2[l(\hat{\kappa}|x) - l(\kappa|x)]$$

which has approximately the chi-square distribution with 1 degree of freedom, denoted as χ_1^2. Then, the $100(1-a)$ % PL interval for κ is $\left[\hat{\kappa}_l^{(PL)}, \hat{\kappa}_u^{(PL)}\right]$ if it satisfies

$$y(\kappa) = C_{1-a}$$

where C_{1-a} denotes the $100(1-a)$ % percentile of χ_1^2. Note that $y(\kappa) = 3.84$ if the PL interval is formulated at the 95% percentile of χ_1^2.

BAYESIAN METHOD

For considering large τ, the prior is believable, that is

$$p(\theta) = \frac{2}{(\ln^2\tau)^7}$$

which is the fisher information of $\ln\tau$, detailed in Cohen (1951). If it is updated with the likelihood function $L(\theta) = \exp(l(\theta))$, then the posterior distribution of θ can be written as

$$p(\theta|x) \propto (\ln^2\tau)^{-\frac{n}{2}-7}\sum_{i=1}^{n}(x_i-\beta)^{-1}\exp\{(2\ln^2\tau)^{-1}[(n-1)\ln^2\hat{\tau} + n(\ln\hat{\omega}-\ln\omega)^2]\}$$

where $\ln^2\hat{\tau} = (n-1)^{-1}\sum_{i=1}^{n}[\ln(x_i-\beta) - \ln\hat{\omega}]^2$. After that, we obtain the posterior of β is

$$p(\beta|x) \propto \sum_{i=1}^{n}(x_i-\beta)^{-1}\int_0^{\infty}(\ln^2\tau)^{-\frac{n}{2}-7}\exp\left[-\frac{(n-1)\ln^2\hat{\tau}}{2\ln^2\tau}\right]\int_{-\infty}^{\infty}\exp\left[-\frac{n}{2\ln^2\tau}(\ln\hat{\omega}-\ln\omega)^2\right]d\ln\omega\ \ d\ln^2\tau$$

$$\propto \sum_{i=1}^{n}(x_i-\beta)^{-1}\int_0^{\infty}(\ln^2\tau)^{-\frac{n-1}{2}-7}\exp\left[\frac{(n-1)\ln^2\hat{\tau}}{2\ln^2\tau}\right]d\ln^2\tau$$

$$\propto \textstyle\sum_{i=1}^{n}(x_i-\beta)^{-1} \tag{9}$$

It can be seen that $p(\beta|x)$ is not a standard distribution, but $F^{-1}()$ can be computed for it. Inverse sampling requires a random variable drawn from the standard uniform distribution $U(0,1)$, denoted as $u \sim U(0,1)$. Next, this drawing is inverted using $F^{-1}()$. To find the inverse of $p(\beta|x)$. It is needed to solve

$$u = \int_{-r}^{r}\textstyle\sum_{i=1}^{n}(x_i-\beta)^{-1}d\beta$$

and then

$$u + \textstyle\sum_{i=1}^{n} [\ln|x_i - r| - \ln|x_i + r|] = 0$$

For solving r, we use the Adam optimization algorithm. Next, it needs to find the posterior of $\ln^2\tau$, that is

$$p(\ln^2\tau|x) \propto (\ln^2\tau)^{-\frac{n}{2}-7}\exp\left[-\frac{(n-1)\ln^2\hat{\tau}}{2\ln^2\tau}\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{n}{\ln^2\tau}(\ln\hat{\omega}-\ln\omega)^2\right.$$

$$\left.-\textstyle\sum_{i=1}^{n}\ln(x_i-\beta)\right]d\ln\omega \;\; d\beta$$

$$\propto (\ln^2\tau)^{-\frac{n+13}{2}-1}\exp\left[-\frac{(n-1)\ln^2\hat{\tau}}{2\ln^2\tau}\right] \qquad (10)$$

which is the inverse gamma distribution with shape $(n + 13)/2$ and scale $(n-1)\ln^2\hat{\tau}/2$. Furthermore, the posterior distribution of $\ln\omega$ becomes

$$p(\ln\omega|x,\ln^2\tau) \propto \exp\left[-\frac{n}{2\ln^2\tau}(\ln\hat{\omega}-\ln\omega)^2\right]\int_{0}^{\infty}\int_{-\infty}^{\infty}(\ln^2\tau)^{-\frac{n}{2}-7}\exp\left[\frac{(n-1)\ln^2\hat{\tau}}{\ln^2\tau}\right]$$

$$\times \textstyle\sum_{i=1}^{n}(x_i-\beta)^{-1}d\ln^2\tau \;\; d\beta$$

$$\propto \exp\left[-\frac{n}{2\ln^2\tau}(\ln\hat{\omega}-\ln\omega)^2\right] \qquad (11)$$

Then, $\ln\omega$ is normally distributed with mean $\ln\omega$ and variance $n^{-1}\ln^2\tau$. From the Equations 9-11, the posterior probability of κ is expressed as

$$\kappa_p = \frac{1}{2}\ln^2\tau_p - \ln\left[1 + \frac{\beta_p}{\omega_p\sqrt{\exp(\ln^2\tau_p)}}\right]$$

where $\omega_p = p(\omega|x)$, $\ln\tau_p = p(\ln\tau|x)$ and $\beta_p = p(\beta|x)$. Therefore, the $100(1 - a)$ % Bayesian highest density (BHD) interval of κ is $[\kappa_p(a/2), \kappa_p(1 - a/2)]$; where $\kappa_p(a)$ is the $100(a)^{th}$ percentile of κ_p.

CONFIDENCE INTERVAL PERFORMANCE

Monte Carlo evaluation is conducted to assess the performance of the CIs in terms of the coverage rate (CR) and the mean length (ML). It is important to note that the CR represents the proportion of intervals that fail to cover the true CV (κ). The best-performing method produces the least ML and a CR of greater than 0.95. As outlined earlier, the CIs are obtained using four methods: Likelihood-based, parametric bootstrap, profile likelihood, and Bayesian. These methods utilize the MLEs, which are solved using the Adam optimization algorithm. These methodologies are identified as ABCP-Adam, LCI-Adam, PL-Adam, and BHD-Adam. To assess their effectiveness, we compare them with alternative approaches based on the zero-skewness estimate (Griffiths 1980) and estimators derived from Royston's index of skewness (Royston 1992). The comparative approaches are classified as (ABCP-ZS, LCI-ZS, PL-ZS, BHD-ZS) for the zero-skewness methods, and (ABCP-RIS, LCI-RIS, PL-RIS, BHD-RIS) for those based on Royston's index.

Griffiths (1980) introduced the zero-skewness estimator for the threshold parameter, which serves as the lower bound of X and allows for the estimation of the mean and variance parameters. Later, Royston (1992) developed a modified version of the Shapiro-Wilk goodness-of-fit test for normality by employing a data transformation technique and the zero-skewness estimator of the threshold. Therefore, both estimators are compared with Adam's optimization algorithm for estimating (β, ω, lnτ) to formulate CIs for κ using the methods described earlier. Based on the Monte Carlo simulation results presented in Table 1 and Figures 1-2, we draw the following conclusions: 1) The BHD-Adam interval showed excellent performance in both coverage rate and mean length. Therefore, it can be regarded as the best confidence interval, 2) The LCI-Adam shows a well-performing interval in terms of coverage rate, while the coverage rate of LCI-RS meets the criteria except for small sample sizes. The LCI-ZS provides good coverage, particularly for large sample sizes, 3) The PL interval achieves coverage rates exceeding 0.95, indicating superior performance in terms of coverage. However, it has the widest mean length, 4) The PL interval performed quite well, maintaining a valid coverage rate over a wide range of parameter combinations. Therefore, it can be considered as an alternative method, 5) The ABCP interval had a lower coverage rate compared to the others, indicating that it was unable to cover the true parameter, and 6) Given the current state of modern computing power, computations can be performed effectively. Therefore, it is recommended to use the BHD interval based on Adam's estimates in practical applications as well.

REAL EXAMPLE–RAINFALL KINETIC ENERGY DATA

Kinetic energy, as highlighted by Meshesha, Tsunekawa and Haregeweyn (2019), and Salles, Poesen, and Sempere-Torres (2002), serves as an indicator of rain's potential to detach soil and assess the risk of water-induced soil erosion. It is crucial to consider a range of raindrop sizes when studying rainfall kinetic energy. In this particular investigation, raindrop size data was sourced from the Land Development Regional Office 6 in Chiang Mai, Northern Thailand. The measurements were conducted using the flour pellet method, a manual technique for measuring raindrop diameters (Kathiravelu, Lucke & Nichols 2016). The analyzed dataset consists of 373 rain records, with energy values measured in joules per hour (J/hr), as illustrated in Figure 3. For the purpose of our study, we exclude any covariates that could potentially influence rainfall kinetic energy. Our primary objective in this study was to estimate the CV for rainfall kinetic energy.

For determining whether sample data fit a model, we use the Akaike and Bayesian information criteria (AIC and BIC) to evaluate how well a 3PLN model fits the data of rainfall kinetic energy. Both criteria define the

TABLE 1. Coverage rates and mean lengths of 95% 95% CIs for κ when ω = exp (3) and τ = exp (3.25)

Methods	The lower bound of X	Sample sizes					
		n = 30		n = 50		n = 100	
		CR	ML	CR	ML	CR	ML
ABCP-Adam	3	**0.949**	2.954	**0.941**	1.990	0.935	1.220
	5	**0.944**	2.998	**0.941**	2.010	0.936	1.230
	10	**0.948**	3.075	**0.940**	2.069	0.939	1.274
ABCP-RS	3	0.778	1.553	0.821	1.239	0.867	0.908
	5	0.780	1.589	0.821	1.269	0.861	0.914
	10	0.788	1.677	0.826	1.322	0.865	0.955
ABCP-ZS	3	0.775	1.548	0.812	1.240	0.852	0.909
	5	0.778	1.589	0.813	1.270	0.840	0.915
	10	0.783	1.673	0.820	1.322	0.845	0.955
LCI-Adam	3	**0.999**	3.499	**0.997**	2.404	**0.995**	1.541
	5	**0.998**	3.553	**0.999**	2.427	**0.996**	1.557
	10	**0.998**	3.641	**0.998**	2.507	**0.997**	1.615
LCI-RS	3	0.907	2.469	**0.940**	1.918	**0.965**	1.361
	5	0.911	2.528	**0.941**	1.955	**0.965**	1.374
	10	0.920	2.647	**0.946**	2.038	**0.967**	1.434
LCI-ZS	3	0.908	2.458	0.934	1.907	**0.960**	1.353
	5	0.911	2.522	0.933	1.944	**0.954**	1.366
	10	0.913	2.636	0.935	2.023	**0.960**	1.424
PL-Adam	3	**0.965**	3.599	**0.994**	3.387	**0.997**	3.319
	5	**0.960**	3.579	**0.991**	3.370	**0.999**	3.298
	10	**0.999**	5.592	**0.992**	4.737	**0.999**	4.075
PL-RS	3	0.918	3.025	**0.970**	2.887	**0.998**	2.783
	5	**0.941**	3.236	**0.981**	3.089	**0.999**	2.953
	10	**0.960**	3.385	**0.991**	3.219	**0.999**	3.097
PL-ZS	3	0.917	3.001	**0.963**	2.857	**0.996**	2.750
	5	**0.940**	3.223	**0.979**	3.063	**0.999**	2.924
	10	**0.959**	3.363	**0.991**	3.186	**0.999**	3.066
BHD-Adam	3	**0.945**	**1.421**	**0.955**	**1.170**	**0.954**	**0.872**
	5	**0.940**	**1.430**	**0.950**	**1.167**	**0.947**	**0.867**
	10	**0.950**	**1.435**	**0.955**	**1.173**	**0.941**	**0.871**
BHD-RS	3	0.507	0.966	0.633	0.909	0.768	0.757
	5	0.521	0.972	0.641	0.910	0.767	0.750
	10	0.555	0.979	0.684	0.911	0.809	0.752
BHD-ZS	3	0.511	0.961	0.639	0.903	0.760	0.752
	5	0.539	0.969	0.647	0.904	0.762	0.745
	10	0.568	0.975	0.684	0.903	0.802	0.746

Results are based on 5,000 replications. The average standard errors for CR and ML are 0.0121 and 0.1002, respectively. Values in bold indicate the best-performing method when it appears in both MR and ML

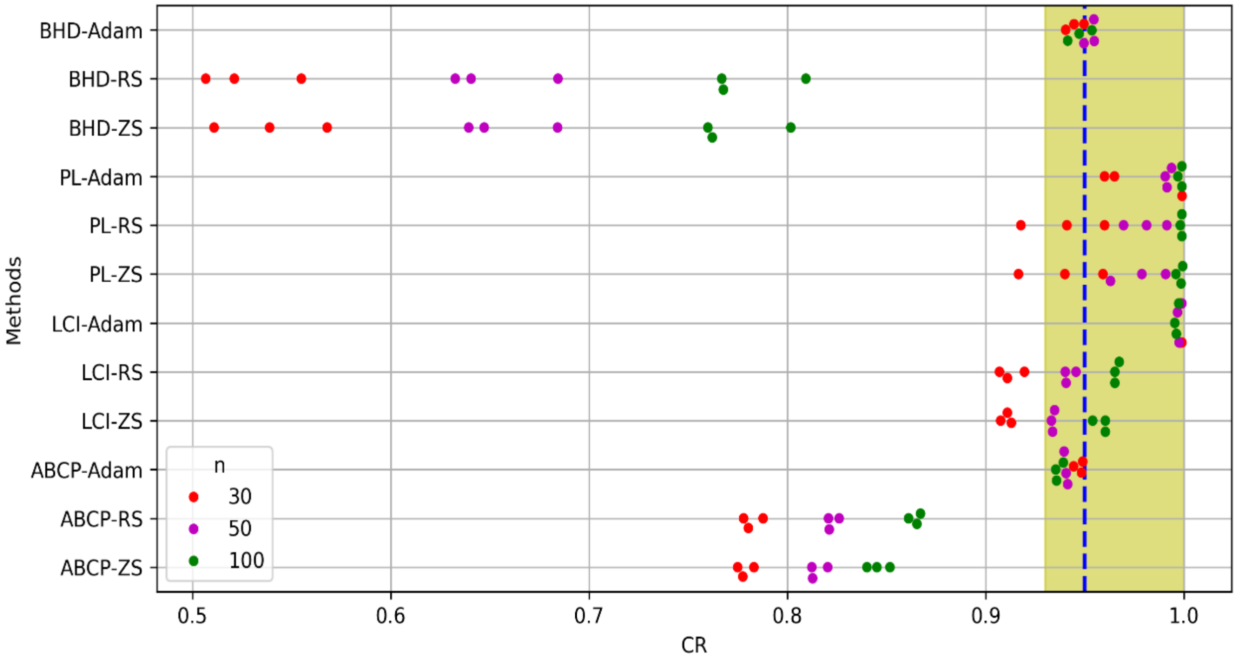

FIGURE 1. Plot of the CR performances for 95% CIs for κ when $\omega = \exp(3)$ and $\tau = \exp(3.25)$

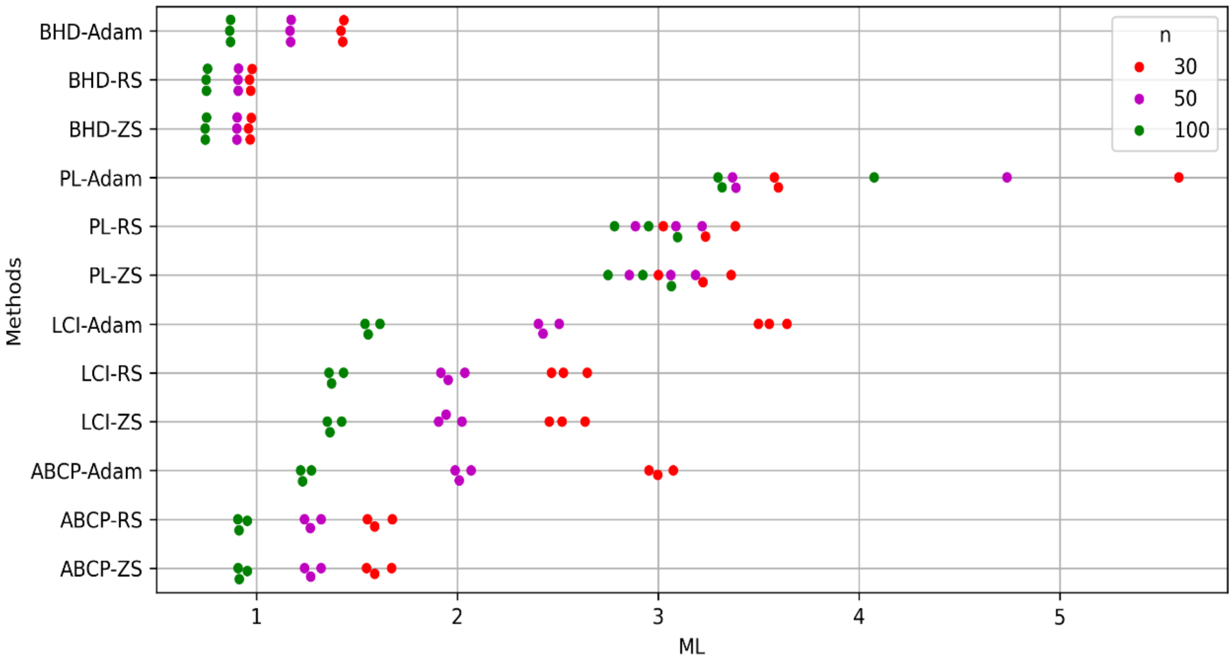

FIGURE 2. Plot of the ML performances for 95% CIs for κ when $\omega = \exp(3)$ and $\tau = \exp(3.25)$

relative information value of the model using the MLE and the number of parameters in the model, defined by AIC = 2K − 2ln(L) and BIC = Kln(n) − 2ln (L), where K is the number of parameters used and L is the likelihood estimate, and n is the number of observations. To determine the suitability of the data model, we compare the AIC and BIC for various models, including Cauchy, chi-square, exponential, gamma, 3PLN, lognormal, logistic, normal, Student's t, and Weibull. Notably, the AIC and BIC values for the 3PLN model are the lowest among all the models considered. This indicates that the 3PLN model is the most efficient and appropriate model for estimating rainfall kinetic energy in the study area. You can find the detailed AIC and BIC results in Table 2.

Based on the given observations, the mean rainfall kinetic energy is calculated as 752.4687 J/hr, and the variance is determined to be 683.7725^2 $(J/hr)^2$. By applying the Adam optimization algorithm, we obtain the parameter estimates $\hat{\beta}$ = 5.2971, $\hat{\omega}$ = 551.1974, and $\hat{\tau}$ = 2.1815. This is further supported by a CV value of exp $(\hat{\kappa})$ = exp(0.2971) =1.346, which indicates relatively minor variability (Asfaw et al. 2018) in the data on the kinetic energy of rainfall. However, if high variation were present, it would suggest that rainfall events in the study area experience significant fluctuations in erosive potential, ranging from very low to very high energy levels. Consequently, soil conservation strategies would need to be designed to address both typical rainfall conditions and extreme events that could potentially cause severe soil erosion.

The threshold parameter is estimated as $\hat{\beta}_{ZS}$ = −37.3109 using the zero-skewness estimator proposed by Griffiths (1980), and $\hat{\beta}_{RIS}$ = −40.8049 using Royston's index of skewness estimator (Royston 1992). However, both of these estimates do not satisfy the condition $0 \leq \beta < \infty$, as elaborated by Pang et al. (2005). Consequently, the 95% CIs for κ based on these two estimators are not considered.

When computing interval estimators, 95% CIs for κ are calculated using three different methods, and all of these intervals include the point estimator $\hat{\kappa}$ (Figure 4). Based on the numerical results obtained for a large sample size of n = 100 and a relatively large threshold β = 10, the BHD-Adam interval demonstrates a valid coverage rate and the narrowest mean length. Therefore, it can be considered the most suitable method for formulating CIs for κ.

DISCUSSION

This paper aims to evaluate the performance of three methods for estimating confidence intervals of the CV in a 3PLN distribution, particularly focusing on cases with a large variation τ = exp (3.25). Our results suggest that the profile-likelihood and both of likelihood-based and Bayesian-based Adam optimization consistently provides valid CIs in terms of the coverage rate. When considering both the coverage rate and the mean length, the Bayesian method based on Adam optimizer is the most suitable. However, it should be noted that due to the highly skewed nature of the data, estimating the threshold using the zero-skewness estimator (Griffiths 1980) and Royston's index of skewness estimator (Royston 1992) has limitations, as shown in Real Example section. Royston's estimator is similar to the zero-skewness method, but it employs a different measure of skewness. Moreover, the Adam optimization algorithm proves to be a feasible approach in the context of this study.

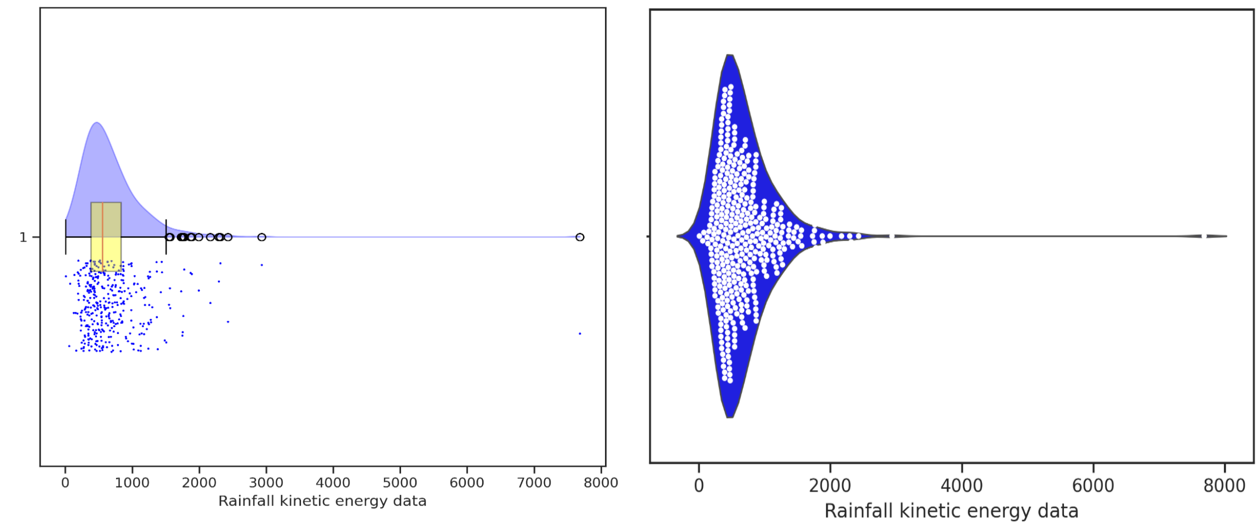

FIGURE 3. Density plots of rainfall kinetic energy collected in Chiang Mai during May-October 2021

TABLE 2. The AIC and BIC values for ten related models

Models	AIC	BIC
Cauchy	5553.232	5561.075
Chi-square	158074.2	158078.1
Exponential	5620.389	5624.310
Gamma	5461.338	5469.181
Three-parameter lognormal	**5423.187**	**5434.952**
Lognormal	5455.285	5463.129
Logistic	5563.345	5571.188
Normal	5767.907	5775.750
Student's t	5503.959	5515.723
Weibull	5517.059	5524.902

Bold represents the smallest AIC and BIC, indicating the most appropriate model

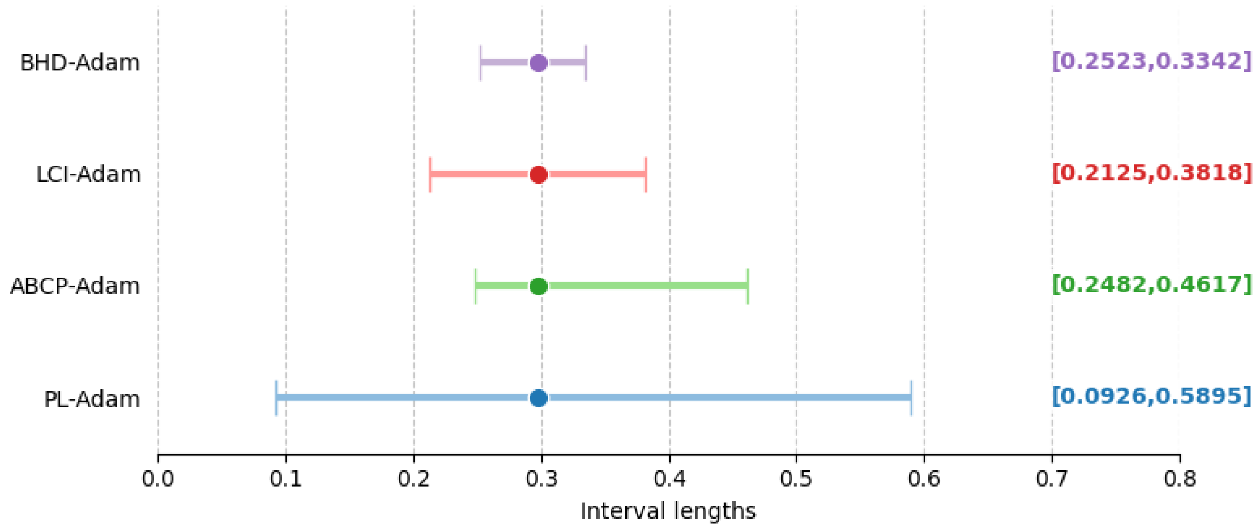

FIGURE 4. 95% confidence limits for the log-transformed CV of the kinetic energy density of rainfall for three methods based on Adam optimization

The likelihood-based interval, which utilizes estimator derived by the Royston (1992), is the next best option for formulating confidence intervals in the context of the 3PLN model. However, it should be noted that this method may have unacceptable coverage when dealing with small sample size. This issue may arise due to the influence of the normal approximation obtained through the central limit theorem.

The ABCP interval, generated through the parametric bootstrap method, provide a narrower coverage rate compared to other methods, particularly in Royston's and zero-skewness estimators. This behavior aligns with the findings of Hesterberg (2015) and Ialongo (2019). In the case of skewed data, it is reasonable to expect lower coverage performance due to the challenge of accurately estimating skewness with limited sample sizes. It is worth noting that skewness is defined as a measure of distribution asymmetry and can impact CI issues (Hesterberg 2015). It is possible that the bootstrap percentile interval efficiently cover the true parameter when the data are drawn from a symmetrical distribution.

Throughout this paper, the performance of three methods assumes that the data follows a 3PLN distribution. However, in certain situations, the observed data may be better modeled by alternative distributions, such as the lognormal (Aitchison & Brown 1963) or gamma (Stacy & Mihram 1965) distributions. To assess the robustness of the methods to depart from the 3PLN model, further research is necessary. This issue has been explored in previous studies by Frey and Zhang (2021) and Moustafa Omar and Aamir (2020).

CONCLUSIONS

Our contribution to this study is the presentation of four statistical methods to construct CIs for the CV of a 3PLN model. The four methods employed are the likelihood-based parametric bootstrap, profile likelihood, and Bayesian approaches. In order to estimate the parameters of the 3PLN model, we utilize Adam's optimization algorithm and compare its performance with that of the zero-skewness estimator (Griffiths 1980) and Royston's index of skewness estimator (Royston 1992).

In situations with overdispersed data, the three-parameter lognormal model stands out as the preferred option for parameter estimation and constructing confidence intervals for the coefficient of variation. Among the four methods assessed in this investigation, the Bayesian approach, based on confidence intervals and utilizing the Adam optimizer, is suggested for scenarios encompassing a wide range of variation, from small to large.

In Table 1 and Figures 1-2, we observe that all four methods provide 95% confidence limits for the CV that cover the true parameter, with the LCI interval yielding the second narrowest length. However, it is important to note that the LCI interval exhibits lower coverage performance due to its limitations in handling highly skewed models. Therefore, based on our findings, we suggest utilizing the CI-based Bayesian method-based Adam optimization algorithm for constructing CIs for the CV, particularly when dealing with small to large variation.

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